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# POROSITY IN THE SPACE OF HÖLDER-FUNCTIONS.

MOHAMMED BACHIR

ABSTRACT. Let  $(X, d)$  be a bounded metric space with a base point  $0_X$ ,  $(Y, \|\cdot\|)$  be a Banach space and  $\text{Lip}_0^\alpha(X, Y)$  be the space of all  $\alpha$ -Hölder-functions that vanish at  $0_X$ , equipped with its natural norm ( $0 < \alpha \leq 1$ ). Let  $0 < \alpha < \beta \leq 1$ . We prove that  $\text{Lip}_0^\beta(X, Y)$  is a  $\sigma$ -porous subset of  $\text{Lip}_0^\alpha(X, Y)$ , if (and only if)  $\inf\{d(x, x') : x, x' \in X; x \neq x'\} = 0$  (i.e.  $d$  is non-uniformly discrete). A more general result will be given.

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**Keyword, phrase:** vector-valued Lipschitz and Hölder-functions, vector-valued Linear operators,  $\sigma$ -porosity, barrier cone.

## 1. INTRODUCTION

The main result of this note is Theorem 1, which gives a condition for some class of subsets of Lipschitz functions to be  $\sigma$ -porous subsets. The result in the abstract, as well as all the other results of this note, are just a very immediate consequence of this main result. However, the main motivations which led to the main theorem of this note, was precisely the result mentioned in the abstract.

Given a metric space  $(X, d)$  with a distinguished point  $0_X$  (called a base point of  $X$ ) and a Banach space  $(Y, \|\cdot\|)$ , we denote by  $\text{Lip}_0(X_d, Y)$  (or by  $\text{Lip}_0(X, Y)$ , if no ambiguity arises) the Banach space of all Lipschitz functions from  $X$  into  $Y$  that vanish at the base point  $0_X$ , equipped with its natural norm defined by

$$\|f\|_L := \sup\left\{\frac{\|f(x) - f(x')\|}{d(x, x')} : x, x' \in X; x \neq x'\right\}, \forall f \in \text{Lip}_0(X_d, Y).$$

We denote simply  $\text{Lip}_0(X_d)$  or  $\text{Lip}_0(X)$ , if  $Y = \mathbb{R}$ . The space  $L(X, Y)$  denotes the space of all linear bounded operators from  $X$  into  $Y$ . The space  $X^*$  denotes the topological dual of  $X$ . Notice that the space  $\text{Lip}_0(X, Y)$  can be isometrically identified to  $L(\mathcal{F}(X), Y)$  where  $\mathcal{F}(X)$  is the free-Lipschitz space over  $X$  introduced by Godefroy-Kalton in [2]. Let us recall the definition of  $\sigma$ -porosity.

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**Definition 1.** Let  $(F, d)$  be a metric space and  $A$  be a subset of  $F$ . A set  $A$  of  $F$  is called porous if there is a  $c \in (0, 1)$  so that for every  $x \in A$  there are  $(y_n) \subset F$  with  $y_n \rightarrow x$  and so that  $B(y_n, cd(y_n, x)) \cap A = \emptyset$  for every  $n$  (We denote by  $B(z, r)$  the closed ball with center  $z$  and radius  $r$ ). A set  $A$  is called  $\sigma$ -porous if it can be represented as a union  $A = \cup_{n=0}^{+\infty} A_n$  of countably many porous sets (the porosity constant  $c_n$  may vary with  $n$ ).

Every  $\sigma$ -porous set is of first Baire category. Moreover, in  $\mathbb{R}^n$ , every  $\sigma$ -porous set is of Lebesgue measure zero. However, there does exist a non- $\sigma$ -porous subset of  $\mathbb{R}^n$  which is of the first category and of Lebesgue measure zero (for more informations about  $\sigma$ -porosity, we refer to [8] and [6]).

**The property  $(\mathcal{P})$ .** Let  $(X, d)$  be a metric space and  $Y$  be a Banach space. Let  $F$  be a nonempty (closed) convex cone of  $\text{Lip}_0(X, Y)$ . We say that  $F$  satisfies property  $(\mathcal{P})$  if there exists a positive constant  $K_F > 0$  depending only on  $F$  such that:

$$(\mathcal{P}) \quad \forall (x, x') \in X \times X, \exists p \in F : \|p\|_L \leq K_F \text{ and } \|p(x) - p(x')\| = d(x, x').$$

This property is related to the Hahn-Banach theorem and norming sets.

*Examples 1.* The property  $(\mathcal{P})$  satisfied in the following cases:

(i) if  $X$  is a normed space and  $F$  contains the space  $X^*.e := \{x \mapsto p(x).e : p \in X^*\}$ , where  $e \in Y$  is a fixed point such that  $\|e\| = 1$ .

(ii) if  $(X, d)$  is a metric space and  $F$  contains the functions  $d_z : x \mapsto d(x, z).e$ , for all  $z \in X$ , where  $e \in Y$  is a fixed point is such that  $\|e\| = 1$ .

(iii) In particular, the space  $\text{Lip}_0(X, Y)$  satisfies the property  $(\mathcal{P})$ . If moreover,  $X$  is a normed space, then  $L(X, Y)$  has the property  $(\mathcal{P})$  too.

*Proof.* (i) By the Hahn-Banach theorem, for all  $x \in X$  there exists  $x^* \in X^*$  such that  $\|x^*\| = 1$  and  $x^*(x) = \|x\|$ . Then, for each  $x \in X$ , we consider the continuous linear map  $p_x = x^*.e : X \rightarrow Y$  defined by  $p_x(z) = x^*(z)e$  for all  $z \in X$ , and the property  $(\mathcal{P})$  is satisfied.

(ii) Immediat.

(iii) This part follows from (i) and (ii) respectively.  $\square$

## 2. THE MAIN RESULT

We are going to give the proof of the main result of this note. Let  $(X, d)$  be a metric space with a base point 0 and  $Y$  be a Banach space. Let  $F \subset \text{Lip}_0(X, Y)$  and  $\phi : X \times X \rightarrow \mathbb{R}^+$  be a positive function such that  $\phi(x, x') = 0$  if and only if  $x = x'$ . For each real number  $s > 0$ , we denote:

$$\mathcal{N}_{\phi, s}(F) := \{f \in F : \sup_{x, x' \in X; x \neq x'} \frac{\|f(x) - f(x')\|}{\phi(x, x')} \leq s\},$$

$$\mathcal{N}_{\phi}(F) := \{f \in F : \sup_{x, x' \in X; x \neq x'} \frac{\|f(x) - f(x')\|}{\phi(x, x')} < +\infty\}.$$

Notice that  $\mathcal{N}_\phi(F) = \cup_{k \in \mathbb{N}} \mathcal{N}_{\phi,k}(F)$  and  $\mathcal{N}_\psi(F) \subset \mathcal{N}_\phi(F)$  if  $\psi \leq \phi$ .

**Theorem 1.** *Let  $F$  be a nonempty (closed) convex cone of  $\text{Lip}_0(X, Y)$  satisfying  $(\mathcal{P})$ . Let  $\phi : X \times X \rightarrow \mathbb{R}^+$  be any positive function such that  $\phi(x, x') = 0$  if and only if  $x = x'$ . Suppose that  $\inf\{\frac{\phi(x, x')}{d(x, x')} : x, x' \in X; x \neq x'\} = 0$ , then for every positive real number  $s > 0$ , we have that  $\mathcal{N}_{\phi,s}(F)$  is a porous subset of  $(F, \|\cdot\|_L)$ . Consequently, the following assertions are equivalent.*

- (1)  $\mathcal{N}_\phi(F) \neq F$ .
- (2)  $\inf\{\frac{\phi(x, x')}{d(x, x')} : x, x' \in X; x \neq x'\} = 0$ .
- (3)  $\mathcal{N}_\phi(F)$  is a  $\sigma$ -porous subset of  $(F, \|\cdot\|_L)$ .

*Proof.* (1)  $\implies$  (2). Suppose that  $\alpha := \inf\{\frac{\phi(x, x')}{d(x, x')} : x, x' \in X; x \neq x'\} > 0$ , then  $\phi(x, x') \geq \alpha d(x, x')$  for all  $x, x' \in X$ . It follows that for every  $f \in F$ , we have that

$$\sup_{x, x' \in X; x \neq x'} \frac{\|f(x) - f(x')\|}{\phi(x, x')} \leq \|f\|_L \sup_{x, x' \in X; x \neq x'} \frac{d(x, x')}{\phi(x, x')} \leq \frac{\|f\|_L}{\alpha} < +\infty.$$

Thus,  $\mathcal{N}_\phi(F) = F$ . Part (3)  $\implies$  (1) is trivial.

Let us prove that if  $\inf\{\frac{\phi(x, x')}{d(x, x')} : x, x' \in X; x \neq x'\} = 0$ , then for every  $s > 0$ , we have that  $\mathcal{N}_{\phi,s}(F)$  is a porous subset of  $(F, \|\cdot\|_L)$ , this gives in particular (2)  $\implies$  (3). Indeed, if  $\inf\{\frac{\phi(x, x')}{d(x, x')} : x, x' \in X; x \neq x'\} = 0$ , then there exists a pair of sequences  $(a_n), (b_n) \subset X$  such that  $0 < r_n := \frac{\phi(a_n, b_n)}{d(a_n, b_n)} \rightarrow 0$ . By assumption, there exists  $K_F > 0$  and a sequence  $(p_n) \subset F$  such that  $\|p_n\|_L \leq K_F$  and  $\|p_n(a_n) - p_n(b_n)\| = d(a_n, b_n)$ , for all  $n \in \mathbb{N}$ . Let  $f \in \mathcal{N}_{\phi,s}(F)$ , then we have that  $\sup_{x, x' \in X; x \neq x'} \frac{\|f(x) - f(x')\|}{\phi(x, x')} \leq s$ . It follows that

$$\begin{aligned} \frac{\|(f + \sqrt{r_n} p_n)(a_n) - (f + \sqrt{r_n} p_n)(b_n)\|}{\phi(a_n, b_n)} &\geq \sqrt{r_n} \frac{\|p_n(a_n) - p_n(b_n)\|}{\phi(a_n, b_n)} - \frac{\|f(a_n) - f(b_n)\|}{\phi(a_n, b_n)} \\ &\geq \sqrt{r_n} \frac{d(a_n, b_n)}{\phi(a_n, b_n)} - \sup_{x, x' \in X; x \neq x'} \frac{\|f(x) - f(x')\|}{\phi(x, x')} \\ &\geq \frac{1}{\sqrt{r_n}} - s \end{aligned}$$

Since,  $r_n \rightarrow 0$ , when  $n \rightarrow +\infty$ , there exists a subsequence  $(r_{n_m})$  such that

$$\frac{1}{\sqrt{r_{n_m}}} > 4s, \quad \forall m \in \mathbb{N}.$$

We set  $f_m = f + \sqrt{r_{n_m}} p_{n_m} \in F$ , for all  $m \in \mathbb{N}$ . We have that

$$\|f_m - f\|_L = \sqrt{r_{n_m}} \|p_{n_m}\|_L \leq K_F \sqrt{r_{n_m}} \rightarrow 0 \text{ when } m \rightarrow +\infty.$$

Let us prove that  $B(f_m, \frac{1}{2K_F}\|f_m - f\|_L) \subset F \setminus \mathcal{N}_{\phi,s}(F)$  for all  $m \in \mathbb{N}$ . Indeed, let  $g \in B(f_m, \frac{1}{2}\|f_m - f\|_L)$ , then we have using the above informations that

$$\begin{aligned}
\frac{\|g(a_{n_m}) - g(b_{n_m})\|}{\phi(a_{n_m}, b_{n_m})} &\geq \frac{\|f_m(a_{n_m}) - f_m(b_{n_m})\|}{\phi(a_{n_m}, b_{n_m})} - \frac{\|(f_m - g)(a_{n_m}) - (f_m - g)(b_{n_m})\|}{\phi(a_{n_m}, b_{n_m})} \\
&\geq \left(\frac{1}{\sqrt{r_{n_m}}} - s\right) - \|f_m - g\|_L \frac{d(a_{n_m}, b_{n_m})}{\phi(a_{n_m}, b_{n_m})} \\
&\geq \left(\frac{1}{\sqrt{r_{n_m}}} - s\right) - \frac{1}{2K_F} \|f_m - f\|_L \frac{d(a_{n_m}, b_{n_m})}{\phi(a_{n_m}, b_{n_m})} \\
&\geq \left(\frac{1}{\sqrt{r_{n_m}}} - s\right) - \frac{1}{2\sqrt{r_{n_m}}} \frac{1}{r_{n_m}} \\
&= \frac{1}{2\sqrt{r_{n_m}}} - s \\
&> s.
\end{aligned}$$

Thus, we have that  $g \in F \setminus \mathcal{N}_{\phi,s}(F)$  and so that  $B(f_m, \frac{1}{2}\|f_m - f\|_L) \subset F \setminus \mathcal{N}_{\phi,s}(F)$  for all  $m \in \mathbb{N}$ . Thus,  $\mathcal{N}_{\phi,s}(F)$  is porous in  $F$  (with  $c = \frac{1}{2K_F}$ ). It follows that  $\mathcal{N}_\phi(F) = \cup_{k \in \mathbb{N}} \mathcal{N}_{\phi,k}(F)$  is  $\sigma$ -porous in  $(F, \|\cdot\|_L)$ .  $\square$

**2.1. Immediate consequences.** We deduce immediately the result mentioned in the abstract.

**Corollary 1.** *Let  $X_1 := (X, d_1)$  and  $X_2 := (X, d_2)$  be a set equipped with two metrics such that  $d_1 \leq d_2$  and let  $(Y, \|\cdot\|)$  be a Banach space. Then,  $\text{Lip}_0(X_1, Y)$  is a  $\sigma$ -porous subset of  $\text{Lip}_0(X_2, Y)$  if (and only if)  $d_1$  and  $d_2$  are not equivalent, if and only if  $\text{Lip}_0(X_1, Y) \neq \text{Lip}_0(X_2, Y)$ .*

*Proof.* We use Theorem 1 and part (iii) of Exemple 1 observing the following equality  $\text{Lip}_0(X_1, Y) = \mathcal{N}_{d_1}(\text{Lip}_0(X_2, Y))$ .  $\square$

Notice that if  $0 < \alpha \leq 1$  and  $d$  is a metric, so is  $d^\alpha$ , hence the above corollary applies to the space of  $\alpha$ -Hölder-functions that vanish at  $0_X$  which is  $\text{Lip}_0^\alpha(X, Y) := \text{Lip}_0(X_{d^\alpha}, Y)$ . Notice also that if  $0 < \alpha < \beta \leq 1$  and  $d$  is bounded, then  $\text{Lip}_0^\beta(X, Y) \subset \text{Lip}_0^\alpha(X, Y)$ . The metrics  $d^\alpha$  and  $d^\beta$  are not equivalent if and only if,  $\inf\{\frac{d^\beta(x, x')}{d^\alpha(x, x')} : x, x' \in X; x \neq x'\} = 0$ , if and only if  $\inf\{d(x, x') : x, x' \in X; x \neq x'\} = 0$  (since  $\beta > \alpha$ ). Thus, we get the result of the abstract.

**Corollary 2.** *Let  $(X, d)$  be a bounded metric space with a base point  $0_X$ ,  $(Y, \|\cdot\|)$  be a Banach space and  $0 < \alpha < \beta \leq 1$ . Then,  $\text{Lip}_0^\beta(X, Y)$  is a  $\sigma$ -porous subset of  $\text{Lip}_0^\alpha(X, Y)$ , if and only if  $\inf\{d(x, x') : x, x' \in X; x \neq x'\} = 0$ .*

Similarly to the case of lipschitz spaces, we obtain the following results in the linear case.

**Corollary 3.** *Let  $X_1 := (X, \|\cdot\|_1)$  and  $X_2 := (X, \|\cdot\|_2)$  be a linear space equipped with two norms such that  $\|\cdot\|_1 \leq \|\cdot\|_2$  and let  $(Y, \|\cdot\|)$  be a Banach space. Then,  $L(X_1, Y)$  is a  $\sigma$ -porous subset of  $L(X_2, Y)$  if and only if  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are not equivalent if and only if  $L(X_1, Y) \neq L(X_2, Y)$ .*

*Proof.* We use Theorem 1 and part (iii) of Exemple 1 after observing that  $L(X_1, Y) = \mathcal{N}_{\|\cdot\|_1}(L(X_2, Y))$ .  $\square$

*Example 1.* Let  $i : (l^1(\mathbb{N}), \|\cdot\|_1) \rightarrow (l^1(\mathbb{N}), \|\cdot\|_\infty)$  be the continuous identity map. Then the image of the adjoint  $i^*$  of  $i$  is a  $\sigma$ -porous subset of  $(l^\infty(\mathbb{N}), \|\cdot\|_\infty)$ .

We give in the following corollary a connexion between the surjectivity of the adjoint  $T^*$  of a one-to-one bounded linear operator  $T$  and the non- $\sigma$ -porosity of its image (see in this sprit, the open mapping theorem in [7, Theorem 2.11]).

**Proposition 1.** *Let  $(X, \|\cdot\|_X)$  and  $(Z, \|\cdot\|_Z)$  be Banach spaces. Let  $T : X \rightarrow Z$  be a one-to-one bounded linear operator and  $T^*$  its adjoint. Then, the following assertions are equivalent.*

- (i)  $T^*(Z^*)$  is not a  $\sigma$ -porous subset of  $X^*$ .
- (ii) There exists  $\alpha > 0$  such that  $\alpha\|x\|_X \leq \|T(x)\|_Z$  for all  $x \in X$ .
- (iii)  $T^*$  is onto.

*Proof.* Since  $T : X \rightarrow Z$  is a one-to-one bounded linear operator, then, the following map define another norm on  $X$ :

$$\|x\| := \frac{\|T(x)\|_Z}{\|T\|} \leq \|x\|_X, \quad \forall x \in X.$$

Let us denote  $X_1 := (X, \|\cdot\|)$ . By Corollary 3, applied with  $Y = \mathbb{R}$ , we have that  $X_1^*$  is a  $\sigma$ -porous subset of  $X^*$  if and only if  $\|\cdot\|$  and  $\|\cdot\|_X$  are not equivalent. Thus, if (ii) is not satisfied (that is,  $\|\cdot\|$  and  $\|\cdot\|_X$  are not equivalent) then, since  $T^*(Z^*) \subset X_1^*$  we get that  $T^*(Z^*)$  is contained in a  $\sigma$ -porous subset of  $X^*$ . Hence, (i)  $\implies$  (ii) is proved. Now, suppose that (ii) holds, it follows that  $T(X)$  is closed in  $Z$ . Let  $x^* \in X^*$  and define  $\phi$  on  $T(X)$  by  $\phi(T(x)) := x^*(x)$  for all  $x \in X$ . Clearly  $\phi$  is well defined (since  $T$  is one-to-one) and linear continuous on  $T(X)$ . Thus,  $\phi$  extends to a linear continuous functional  $y^* \in Z^*$  and we have  $T^*(y^*) = y^* \circ T = x^*$ . Hence,  $T^*$  is onto and (ii)  $\implies$  (iii) is proved. Part (iii)  $\implies$  (i), is trivial.  $\square$

Let  $(X, \|\cdot\|)$  be a normed space, and let  $S$  be a nonempty subset of the dual space  $X^*$ . The set  $S$  is called separating if:  $x^*(x) = 0$  for all  $x^* \in S$  implies that  $x = 0$ . It is called norming if the functional

$$N_S(x) = \sup_{x^* \in S; x^* \neq 0} \frac{|x^*(x)|}{\|x^*\|},$$

is an equivalent norm on  $X$  (see [2] for the use of this notion).

**Proposition 2.** *Let  $(X, \|\cdot\|)$  be a normed space. Every separating subset  $S \subset X^*$  which is not a  $\sigma$ -porous subset of  $X^*$ , is norming.*

*Proof.* It is clear that  $N(x) \leq \|x\|$  for all  $x \in X$ . On the other hand, we have that

$$S \subset \mathcal{N}_{N_S}(X) := \{x^* \in X^* : \sup_{N_S(x)=1} |x^*(x)| < +\infty\}.$$

Since  $S$  is not contained in a  $\sigma$ -porous subset of  $X^*$ , then  $\mathcal{N}_\phi(X)$  must be non- $\sigma$ -porous, which implies from Theorem 1 that  $\inf_{\|x\|=1} N_S(x) > 0$ . Hence  $N_S$  is equivalent to  $\|\cdot\|$ .  $\square$

**2.2. Coarse Lipschitz function and Lipschitz-free space.** Given a metric space  $(X, d)$  with a base point  $0_X$ , the free space  $\mathcal{F}(X)$  is constructed as follows: we first consider as pivot space the Banach space  $(\text{Lip}_0(X), \|\cdot\|_L)$  of real-valued Lipschitz functions vanishing at the base point. Then each  $x \in X$  is identified to a Dirac measure  $\delta_x$  acting linearly on  $\text{Lip}_0(X)$  as evaluation. Then the mapping

$$\begin{aligned} \delta_X : X &\rightarrow \text{Lip}_0(X)^* \\ x &\mapsto \delta_x \end{aligned}$$

that maps  $x$  to  $\delta_x$  is an isometric embedding. The Lipschitz-free space  $\mathcal{F}(X)$  over  $X$  is defined as the closed linear span of  $\delta(X)$  in  $\text{Lip}_0(X)$ . Furthermore, the free space is a predual for  $\text{Lip}_0(X)$ , meaning that  $\mathcal{F}(X)^*$  is isometrically isomorphic to  $\text{Lip}_0(X)$ . Let  $(X, d)$  and  $(Y, d')$  be two metric spaces, each one with a base point ( $0_X$  and  $0_Y$ , respectively) and  $F : X \rightarrow Y$  a Lipschitz function such that  $R(0_X) = 0_Y$ . Then, it is well known (see [2, Lemma 2.2]) that there exists a unique linear operator  $\widehat{F} : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$  such that  $\|F\|_L = \|\widehat{F}\|$  and  $\delta_Y \circ F = \widehat{F} \circ \delta_X$ . The adjoint of  $\widehat{F}$ , namely  $\widehat{F}^* : \text{Lip}_0(Y) \rightarrow \text{Lip}_0(X)$ , satisfies  $\widehat{F}^*(f) = f \circ F$  for all  $f \in \text{Lip}_0(Y)$ .

A map  $F : (X, d) \rightarrow (Y, d')$  is said to be a coarse Lipschitz, if there exist  $\alpha, \beta > 0$  such that

$$\alpha d(x, x') \leq d'(F(x), F(x')) \leq \beta d(x, x'), \quad \forall x, x' \in X.$$

Combining Proposition 1 together with a similar proof, we obtain in the following proposition, a characterization of coarse Lipschitz maps.

**Proposition 3.** *Let  $(X, d)$  and  $(Y, d')$  be metric spaces with base points  $0_X$  and  $0_Y$  respectively and let  $F : (X, d) \rightarrow (Y, d')$  be a one-to-one Lipschitz map such that  $F(0_X) = 0_Y$ . Then the following assertions are equivalent.*

- (i) *The image of  $\widehat{F}^*$  is not  $\sigma$ -porous in  $\text{Lip}_0(X)$ .*
- (ii) *The map  $F$  is coarse Lipschitz.*
- (iii) *The adjoint  $\widehat{F}^*$  is onto.*
- (iv) *The linear map  $\widehat{F} : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$  is coarse Lipschitz.*

*Proof.* Since,  $F$  is one-to-one, we define the following metric on  $X$

$$d_1(x, x') := \frac{1}{L_F} d'(F(x), F(x')) \leq d(x, x'), \quad \forall x, x' \in X,$$

where  $L_F$  denotes the constant of Lipschitz of  $F$ . Suppose that  $F$  is not coarse Lipschitz, then the metric  $d_1$  is not equivalent to the metric  $d$ . It follows, using Corollary 1, that  $\text{Lip}_0(X_1)$  is  $\sigma$ -porous subset of  $\text{Lip}_0(X)$ , where  $X_1 = (X, d_1)$ . Now, we observe that  $\text{Im}(\widehat{F}^*) := \{f \circ F : f \in \text{Lip}_0(Y)\} \subset \text{Lip}_0(X_1)$ , which implies that  $\text{Im}(\widehat{F}^*)$  is a  $\sigma$ -porous subset of  $(\text{Lip}_0(X), \|\cdot\|_L)$ . Thus, we proved that (i)  $\implies$  (ii). Let us prove that (ii)  $\implies$  (iii). Let  $g \in \text{Lip}_0(X)$ , we need to show that there exists  $f \in \text{Lip}_0(Y)$  such that  $g = f \circ F$ . Indeed, define  $\phi : F(X) \rightarrow \mathbb{R}$  by  $\phi(F(x)) := g(x)$  for all  $x \in X$ . The map  $\phi$  is well defined since  $F$  is one-to-one. On the other hand,  $\phi$  is Lipschitz on  $F(X)$  since  $F$  is coarse Lipschitz. Thus,  $\phi$  extends to a Lipschitz function  $f$  from  $Y$  into  $\mathbb{R}$  with the same constant of Lipschitz, by the inf-convolution formula ( $L_\phi$  denotes the constant of Lipschitz of  $\phi$  on  $F(X)$ ):  $\forall y \in Y$

$$f(y) := \inf\{\phi(y') + L_\phi d(y, y') : y' \in F(X)\}.$$

Hence,  $f \in \text{Lip}_0(Y)$  and  $f \circ F(x) = \phi(F(x)) = g(x)$  for all  $x \in X$  and so (ii)  $\implies$  (iii) is proved. Part (iii)  $\implies$  (i) is trivial. Now, from Proposition 1, we see (iii)  $\iff$  (iv).  $\square$

**2.3. Application to the barrier cone and polar of sets.** Let  $X$  be a normed space and  $K$  be a nonempty subset of  $X$ . The barrier cone of  $K$  is the subset  $B(K)$  of the topological dual  $X^*$  defined by

$$B(K) = \{x^* \in X^* : \sup_{x \in K} x^*(x) < +\infty\}.$$

The polar set of  $K$  is a subset of the barrier cone of  $K$  defined as follows:

$$K^\circ = \{x^* \in X^* : \sup_{x \in K} x^*(x) \leq 1\}.$$

The study of barrier cones has interested several authors. It is shown in [5, Theorem 3.1.1] that for a closed convex subset  $K$  of  $X$ , we have that  $B(K) = X^*$  if and only if  $K$  is bounded, on the other hand,  $B(K)$  is dense in  $X^*$  if and only if  $K$  does not contain any halfline. In general, we know that  $\overline{B(K)} \neq X^*$  (see example in [1]). A study of the closure of the barrier of a closed convex set is given in [1]. As an immediate consequence of main theorem, we obtain below that the barrier cone of some general class of unbounded subsets is a  $\sigma$ -porous subset of the dual  $X^*$ . This shows that in general, the barrier cone may be a "very small" subset of  $X^*$ .

We define the class  $\Phi(X)$  of positive functions on  $X$  (not necessarily continuous) as follows:  $\phi \in \Phi(X)$  if and only if,  $\phi : X \rightarrow \mathbb{R}^+$  and satisfies



- (i)  $\phi(\lambda x) = |\lambda|\phi(x)$  for all  $x \in X$  and all  $\lambda \in \mathbb{R}$
- (ii)  $\phi(x) = 0$  if and only if  $x = 0$

For every  $\phi \in \Phi(X)$ , we denote  $S_\phi := \{x \in X : \phi(x) = 1\}$  and  $C_\phi := \{x \in X : \phi(x) \leq 1\}$ . Notice, that in general  $C_\phi$  is not a convex set (resp. not closed), if we do not suppose that  $\phi$  is a convex function (resp. a continuous function). It is easy to see that

$$(1) \quad C_\phi \text{ is bounded} \iff S_\phi \text{ is bounded} \iff \inf_{x \in X: \|x\|=1} \phi(x) > 0.$$

Thanks to the symmetry of  $\phi \in \Phi(X)$  and the fact that  $B(C_\phi) = B(S_\phi)$ , we have using the notation of Theorem 1, that

$$(2) \quad B(C_\phi) = \mathcal{N}_\phi(X^*).$$

The polar of  $S_\phi$  coincides with  $\mathcal{N}_{\phi,1}(X^*)$  and we have

$$(3) \quad C_\phi^\circ \subset S_\phi^\circ = \mathcal{N}_{\phi,1}(X^*).$$

Now, using (1), (2) and (3) and applying Theorem 1 to the spaces  $F = X^*$  and  $Y = \mathbb{R}$  (using Exemple 1), we get directly the following informations about the size of the barrier cone, as well as the polar of a set of the form  $C_\phi$  in the dual space.

**Corollary 4.** *Let  $X$  be a normed space and  $\phi \in \Phi(X)$ . If  $C_\phi$  is not bounded in  $X$ , then the polar  $C_\phi^\circ$  is contained in a porous subset of  $X^*$ . Moreover, the following assertions are equivalent.*

- (i)  $B(C_\phi) \neq X^*$ .
- (ii)  $C_\phi$  is not bounded in  $X$ .
- (iii)  $B(C_\phi)$  is a  $\sigma$ -porous subset of  $X^*$ .

We deduce that, if  $K$  is any nonempty subset of  $X$  such that  $S_\phi \subset K$ , for some  $\phi \in \Phi(X)$  with  $S_\phi$  not bounded, then the polar  $K^\circ$  is contained in a porous subset of  $X^*$  and the barrier cone  $B(K)$  is contained in a  $\sigma$ -porous subset of  $X^*$ . Notice that if  $K$  is a closed absorbing disk in  $X$ , does not contain a non-trivial vector subspace and is a neighborhood of the origin in  $X$ , then the Minkowski functional  $\phi_K$  of  $K$  is a continuous norm (with respect to the norm  $\|\cdot\|$ , but not equivalent to it, if  $K$  is not bounded) hence  $\phi_K \in \Phi(X)$  and we have that  $K = C_{\phi_K}$ .

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