POROSITY IN THE SPACE OF HÖLDER-FUNCTIONS

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Abstract. Let (X,d) be a bounded metric space with a base point 0_X, $(Y,\|\cdot\|)$ be a Banach space and $\mathrm{Lip}_0^\alpha(X,Y)$ be the space of all α-Hölder-functions that vanish at 0_X, equipped with its natural norm ($0<\alpha\leq 1$). Let $0<\alpha<\beta\leq 1$. We prove that $\mathrm{Lip}_0^\beta(X,Y)$ is a σ-porous subset of $\mathrm{Lip}_0^\alpha(X,Y)$, if (and only if) $\inf\{d(x,x'): x,x'\in X; x\neq x'\}=0$ (i.e. d is non-uniformly discrete). A more general result will be given.

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1. Introduction

The main result of this note is Theorem 1, which gives a condition for some class of subsets of Lipschitz functions to be σ-porous subsets. The result in the abstract, as well as all the other results of this note, are just a very immediate consequence of this main result. However, the main motivations which led to the main theorem of this note, was precisely the result mentioned in the abstract.

Given a metric space (X,d) with a distinguished point 0_X (called a base point of X) and a Banach space $(Y,\|\cdot\|)$, we denote by $\mathrm{Lip}_0(X_d,Y)$ (or by $\mathrm{Lip}_0(X,Y)$, if no ambiguity arises) the Banach space of all Lipschitz functions from X into Y that vanish at the base point 0_X, equipped with its natural norm defined by

$$\|f\|_L := \sup\{\frac{\|f(x)-f(x')\|}{d(x,x')} : x,x'\in X; x\neq x\}, \forall f\in \mathrm{Lip}_0(X_d,Y).$$

We denote simply $\mathrm{Lip}_0(X_d)$ or $\mathrm{Lip}_0(X)$, if $Y=\mathbb{R}$. The space $L(X,Y)$ denotes the space of all linear bounded operators from X into Y. The space X^* denotes the topological dual of X. Notice that the space $\mathrm{Lip}_0(X,Y)$ can be isometrically itentified to $L(\mathcal{F}(X),Y)$ where $\mathcal{F}(X)$ is the free-Lipschitz space over X introduced by Godefroy-Kalton in [2]. Let us recall the definition of σ-porosity.

Definition 1. *Let (F,d) be a metric space and A be a subset of F. A set A of F is called porous if there is a $c \in (0,1)$ so that for every $x \in A$ there are $(y_n) \subset F$ with $y_n \to x$ and so that $B(y_n, cd(y_n,x)) \cap A = \emptyset$ for every n (We denote by $B(z,r)$ the closed ball with center z and radius r). A set A is called σ-porous if it can be represented as a union $A = \cup_{n=0}^{+\infty} A_n$ of countably many porous sets (the porosity constant c_n may vary with n).*

Every σ-porous set is of first Baire category. Moreover, in $\mathbb{R}^n$, every σ-porous set is of Lebesque measure zero. However, there does exist a non-σ-porous subset of $\mathbb{R}^n$ which is of the first category and of Lebesgue measure zero (for more informations about σ-porosity, we refer to [7] and [5]).

The property $(\mathcal{P})$. Let (X,d) be a metric space and Y be a Banach space. Let F be a nonempty (closed) convex cone of $\mathrm{Lip}_0(X,Y)$. We say that F satisfies property $(\mathcal{P})$ if there exists a positive constant $K_F > 0$ depending only on F such that:

$$(\mathcal{P}) \quad \forall (x,x') \in X \times X, \exists p \in F : \|p\|_L \leq K_F \text{ and } \|p(x) - p(x')\| = d(x,x').$$

This property is related to the Hahn-Banach theorem and norming sets.

Examples 1. The property $(\mathcal{P})$ satisfied in the following cases:

(i) if X is a normed space and F contains the space $X^*.e := \{x \mapsto p(x).e : p \in X^*\}$, where $e \in Y$ is a fixed point such that $\|e\| = 1$.

(ii) if (X,d) is a metric space and F contains the functions $d_z : x \mapsto d(x,z).e$, for all $z \in X$, where $e \in Y$ is a fixed point is such that $\|e\| = 1$.

(iii) In particular, the space $\mathrm{Lip}_0(X,Y)$ satisfies the property $(\mathcal{P})$. If moreover, X is a normed space, then $L(X,Y)$ has the property $(\mathcal{P})$ too.

Proof. (i) By the Hahn-Banach theorem, for all $x \in X$ there exists $x^* \in X^*$ such that $\|x^*\| = 1$ and $x^*(x) = \|x\|$. Then, for each $x \in X$, we consider the continuous linear map $p_x = x^*.e : X \to Y$ defined by $p_x(z) = x^*(z)e$ for all $z \in X$, and the property $(\mathcal{P})$ is satisfied.

(ii) Immediat.

(iii) This part follows from (i) and (ii) respectively. $\square$

2. The main result

We are going to give the proof of the main result of this note. Let (X,d) be a metric space with a base point 0 and Y be a Banach space. Let $F \subset \mathrm{Lip}_0(X,Y)$ and $\phi : X \times X \to \mathbb{R}^+$ be a positive function such that $\phi(x,x') = 0$ if and only if $x = x'$. For each real number $s > 0$, we denote:

$$\mathcal{N}_{\phi,s}(F) := \{f \in F : \sup_{x,x' \in X; x \neq x'} \frac{\|f(x) - f(x')\|}{\phi(x,x')} \leq s\},$$

$$\mathcal{N}_{\phi}(F) := \{f \in F : \sup_{x,x' \in X; x \neq x'} \frac{\|f(x) - f(x')\|}{\phi(x,x')} < +\infty\}.$$

Notice that $\mathcal{N}_\phi(F) = \cup_{k\in\mathbb{N}}\mathcal{N}_{\phi,k}(F)$ and $\mathcal{N}_\psi(F) \subset \mathcal{N}_\phi(F)$ if $\psi \leq \phi$.

Theorem 1. *Let F be a nonempty (closed) convex cone of $\mathrm{Lip}_0(X,Y)$ satisfying $(\mathcal{P})$. Let $\phi : X\times X \to \mathbb{R}^+$ be any positive function such that $\phi(x,x') = 0$ if and only if $x = x'$. Suppose that $\inf\{\frac{\phi(x,x')}{d(x,x')} : x, x' \in X; x \neq x'\} = 0$, then for every positive real number $s > 0$, we have that $\mathcal{N}_{\phi,s}(F)$ is a porous subset of $(F, \|\cdot\|_L)$. Consequently, the following assertions are equivalent.*
(1) $\mathcal{N}_\phi(F) \neq F$.
(2) $\inf\{\frac{\phi(x,x')}{d(x,x')} : x, x' \in X; x \neq x'\} = 0$.
(3) $\mathcal{N}_\phi(F)$ *is a σ-porous subset of* $(F, \|\cdot\|_L)$.

Proof. (1) $\Longrightarrow$ (2). Suppose that $\alpha := \inf\{\frac{\phi(x,x')}{d(x,x')} : x, x' \in X; x \neq x'\} > 0$, then $\phi(x,x') \geq \alpha d(x,x')$ for all $x, x' \in X$. It follows that for every $f \in F$, we have that

$$\sup_{x,x'\in X; x\neq x'} \frac{\|f(x) - f(x')\|}{\phi(x,x')} \leq \|f\|_L \sup_{x,x'\in X; x\neq x'} \frac{d(x,x')}{\phi(x,x')} \leq \frac{\|f\|_L}{\alpha} < +\infty.$$

Thus, $\mathcal{N}_\phi(F) = F$. Part (3) $\Longrightarrow$ (1) is trivial.

Let us prove that if $\inf\{\frac{\phi(x,x')}{d(x,x')} : x, x' \in X; x \neq x'\} = 0$, then for every $s > 0$, we have that $\mathcal{N}_{\phi,s}(F)$ is a porous subset of $(F, \|\cdot\|_L)$, this gives in particular (2) $\Longrightarrow$ (3). Indeed, if $\inf\{\frac{\phi(x,x')}{d(x,x')} : x, x' \in X; x \neq x'\} = 0$, then there exists a pair of sequences $(a_n), (b_n) \subset X$ such that $0 < r_n := \frac{\phi(a_n,b_n)}{d(a_n,b_n)} \to 0$. By assumption, there exists $K_F > 0$ and a sequence $(p_n) \subset F$ such that $\|p_n\|_L \leq K_F$ and $\|p_n(a_n) - p(b_n)\| = d(a_n, b_n)$, for all $n \in \mathbb{N}$. Let $f \in \mathcal{N}_{\phi,s}(F)$, then we have that $\sup_{x,x'\in X; x\neq x'} \frac{\|f(x)-f(x')\|}{\phi(x,x')} \leq s$. It follows that

$$\begin{aligned}
\frac{\|(f + \sqrt{r_n}p_n)(a_n) - (f + \sqrt{r_n}p_n)(b_n)\|}{\phi(a_n,b_n)} &\geq \sqrt{r_n}\frac{\|p_n(a_n) - p_n(b_n)\|}{\phi(a_n,b_n)} - \frac{\| - (f(a_n) - f(b_n))\|}{\phi(a_n,b_n)} \\
&\geq \sqrt{r_n}\frac{d(a_n,b_n)}{\phi(a_n,b_n)} - \sup_{x,x'\in X; x\neq x'} \frac{\|f(x) - f(x')\|}{\phi(x,x')} \\
&\geq \frac{1}{\sqrt{r_n}} - s
\end{aligned}$$

Since, $r_n \to 0$, when $n \to +\infty$, there exists a subsequence (r_{n_m}) such that

$$\frac{1}{\sqrt{r_{n_m}}} > 4s, \quad \forall m \in \mathbb{N}.$$

We set $f_m = f + \sqrt{r_{n_m}}p_{n_m} \in F$, for all $m \in \mathbb{N}$. We have that

$$\|f_m - f\|_L = \sqrt{r_{n_m}}\|p_{n_m}\|_L \leq K_F\sqrt{r_{n_m}} \to 0 \text{ when } m \to +\infty.$$

Let us prove that $B(f_m, \frac{1}{2K_F}\|f_m - f\|_L) \subset F \setminus \mathcal{N}_{\phi,s}(F)$ for all $m \in \mathbb{N}$. Indeed, let $g \in B(f_m, \frac{1}{2}\|f_m - f\|_L)$, then we have using the above informations that

$$\begin{aligned}
\frac{\|g(a_{n_m}) - g(b_{n_m})\|}{\phi(a_{n_m}, b_{n_m})} &\geq \frac{\|f_m(a_{n_m}) - f_m(b_{n_m})\|}{\phi(a_{n_m}, b_{n_m})} - \frac{\|(f_m - g)(a_{n_m}) - (f_m - g)(b_{n_m})\|}{\phi(a_{n_m}, b_{n_m})}\\
&\geq (\frac{1}{\sqrt{r_{n_m}}} - s) - \|f_m - g\|_L \frac{d(a_{n_m}, b_{n_m})}{\phi(a_{n_m}, b_{n_m})}\\
&\geq (\frac{1}{\sqrt{r_{n_m}}} - s) - \frac{1}{2K_F}\|f_m - f\|_L \frac{d(a_{n_m}, b_{n_m})}{\phi(a_{n_m}, b_{n_m})}\\
&\geq (\frac{1}{\sqrt{r_{n_m}}} - s) - \frac{1}{2}\sqrt{r_{n_m}}\frac{1}{r_{n_m}}\\
&= \frac{1}{2\sqrt{r_{n_m}}} - s\\
&> s.
\end{aligned}$$

Thus, we have that $g \in F \setminus \mathcal{N}_{\phi,s}(F)$ and so that $B(f_m, \frac{1}{2}\|f_m - f\|_L) \subset F \setminus \mathcal{N}_{\phi,s}(F)$ for all $m \in \mathbb{N}$. Thus, $\mathcal{N}_{\phi,s}(F)$ is porous in F (with $c = \frac{1}{2K_F}$). It follows that $\mathcal{N}_\phi(F) = \cup_{k\in\mathbb{N}}\mathcal{N}_{\phi,k}(F)$ is σ-porous in $(F, \|\cdot\|_L)$. □

2.1. **Immediate consequences.** We deduce immediately the result mentioned in the abstract.

Corollary 1. *Let $X_1 := (X, d_1)$ and $X_2 := (X, d_2)$ be a set equipped with two metrics such that $d_1 \leq d_2$ and let $(Y, \|\cdot\|)$ be a Banach space. Then, $\mathrm{Lip}_0(X_1, Y)$ is a σ-porous subset of $\mathrm{Lip}_0(X_2, Y)$ if (and only if) d_1 and d_2 are not equivalent, if and only if $\mathrm{Lip}_0(X_1, Y) \neq \mathrm{Lip}_0(X_2, Y)$.*

Proof. We use Theorem 1 and part (iii) of Exemple 1 observing the following equality $\mathrm{Lip}_0(X_1, Y) = \mathcal{N}_{d_1}(\mathrm{Lip}_0(X_2, Y))$. □

Notice that if $0 < \alpha \leq 1$ and d is a metric, so is d^α, hence the above corollay applies to the space of α-Hölder-functions that vanish at 0_X which is $\mathrm{Lip}_0^\alpha(X, Y) := \mathrm{Lip}_0(X_{d^\alpha}, Y)$. Notice also that if $0 < \alpha < \beta \leq 1$ and d is bounded, then $\mathrm{Lip}_0^\beta(X, Y) \subset \mathrm{Lip}_0^\alpha(X, Y)$. The metrics d^α and d^β are not equivalent if and only if, $\inf\{\frac{d^\beta(x,x')}{d^\alpha(x,x')} : x, x' \in X; x \neq x'\} = 0$, if and only if $\inf\{d(x, x') : x, x' \in X; x \neq x'\} = 0$ (since $\beta > \alpha$). Thus, we get the result of the abstract.

Corollary 2. *Let (X, d) be a bounded metric space with a base point 0_X, $(Y, \|\cdot\|)$ be a Banach space and $0 < \alpha < \beta \leq 1$. Then, $\mathrm{Lip}_0^\beta(X, Y)$ is a σ-porous subset of $\mathrm{Lip}_0^\alpha(X, Y)$, if and only if $\inf\{d(x, x') : x, x' \in X; x \neq x'\} = 0$.*

For the linear case, we have the following results.

Corollary 3. *Let $X_1 := (X, \|\cdot\|_1)$ and $X_2 := (X, \|\cdot\|_2)$ be a linear space equipped with two norms such that $\|\cdot\|_1 \leq \|\cdot\|_2$ and let $(Y, \|\cdot\|)$ be a Banach space. Then, $L(X_1, Y)$ is a σ-porous subset of $L(X_2, Y)$ if and only if $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent if and only if $L(X_1, Y) \neq L(X_2, Y)$.*

Proof. We use Theorem 1 and part (iii) of Exemple 1 after observing that $L(X_1, Y) = \mathcal{N}_{\|\cdot\|_1}(L(X_2, Y))$. $\square$

Example 1. Let $i : (l^1(\mathbb{N}), \|\cdot\|_1) \to (l^1(\mathbb{N}), \|\cdot\|_\infty)$ be the continuous identity map. Then the image of the adjoint i^* of i is a σ-porous subset of $(l^\infty(\mathbb{N}), \|\cdot\|_\infty)$.

We give in the following corollary a connexion between the surjectivity of the adjoint T^* of a one-to-one bounded linear operator T and the non-σ-porosity of its image (see in this sprit, the open mapping theorem in [6, Theorem 2.11]).

Proposition 1. *Let $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ be Banach spaces. Let $T : X \to Z$ be a one-to-one bounded linear operator and T^* its adjoint. Then, the following assertions are equivalent.*

(i) *$T^*(Z^*)$ is not contained in a σ-porous subset of X^*.*

(ii) *There exists $\alpha > 0$ such that $\alpha\|x\|_X \leq \|T(x)\|_Z$ for all $x \in X$.*

(iii) *T^* is onto.*

Proof. Since $T : X \to Z$ is a one-to-one bounded linear operator, then, the following map define another norm on X:

$$\|x\| := \frac{\|T(x)\|_Z}{\|T\|} \leq \|x\|_X, \quad \forall x \in X.$$

Let us denote $X_1 := (X, \|\cdot\|)$. By Corollary 3, applied with $Y = \mathbb{R}$, we have that X_1^* is a σ-porous subset of X^* if and only if $\|\cdot\|$ and $\|\cdot\|_X$ are not equivalente. Thus, if (ii) is not satisfied (that is, $\|\cdot\|$ and $\|\cdot\|_X$ are not equivalente) then, since $T^*(Z^*) \subset X_1^*$ we get that $T^*(Z^*)$ is contained in a σ-porous subset of X^*. Hence, $(i) \Longrightarrow (ii)$ is proved. Now, suppose that (ii) holds, it follows that $T(X)$ is closed in Y. Let $x^* \in X^*$ and define ϕ on $T(X)$ by $\phi(T(x)) := x^*(x)$ for all $x \in X$. Clearly ϕ is well defined (since T is one-to-one) and linear continuous on $T(X)$. Thus, ϕ extends to a linear continuous functional $y^* \in Y^*$ and we have $T^*(y^*) = y^* \circ T = x^*$. Hence, T^* is onto and $(ii) \Longrightarrow (iii)$ is proved. Part $(iii) \Longrightarrow (i)$, is trivial. $\square$

Let $(X, \|\cdot\|)$ be a normed space, and let S be a nonempty subset of the dual space X^*. The set S is called separating if: $x^*(x) = 0$ for all $x^* \in S$ implies that $x = 0$. It is called norming if the functional

$$N_S(x) = \sup_{x^* \in S; x^* \neq 0} \frac{|x^*(x)|}{\|x^*\|},$$

is an equivalent norm on X (see [2] for the use of this notion).

Proposition 2. *Let $(X, \|\cdot\|)$ be a normed space. Every separating subset $S \subset X^*$ which is not contained in a σ-porous subset of X^*, is norming.*

Proof. It is clear that $N(x) \leq \|x\|$ for all $x \in X$. On the other hand, we have that

$$S \subset \mathcal{N}_{N_S}(X) := \{x^* \in X^* : \sup_{N_S(x)=1} |x^*(x)| < +\infty\}.$$

Since S is not contained in a σ-porous subset of X^*, then $\mathcal{N}_{\phi}(X)$ must be non-σ-porous, which implies from Theorem 1 that $\inf_{\|x\|=1} N_S(x) > 0$. Hence N_S is equivalent to $\|\cdot\|$. □

2.2. **Application to the barrier cone and polar of sets.** Let X be a normed space and K be a nonempty subset of X. The barrier cone of K is the subset $B(K)$ of the topological dual X^* defined by

$$B(K) = \{x^* \in X^* : \sup_{x \in K} x^*(x) < +\infty\}.$$

The polar set of K is a subset of the barrier cone of K defined as follows:

$$K^\circ = \{x^* \in X^* : \sup_{x \in K} x^*(x) \leq 1\}.$$

The study of barrier cones has interested several authors. It is shown in [4, Theorem 3.1.1] that for a closed convex subset K of X, we have that $B(K) = X^*$ if and only if K is bounded, on the other hand, $B(K)$ is dense in X^* if and only if K does not contain any halfline. In general, we know that $\overline{B(K)} \neq X^*$ (see example in [1]). A study of the closure of the barrier of a closed convex set is given in [1]. As an immediat consequence of main theorem, we obtain bellow that the barrier cone of some general class of unbounded subsets is a σ-pourous subset of the dual X^*. This shows that in general, the barrier cone may be a "very small" subset of X^*.

We define the classe $\Phi(X)$ of positive functions on X (not necessarily continuous) as follows: $\phi \in \Phi(X)$ if and only if, $\phi : X \to \mathbb{R}^+$ and satisfies
(*i*) $\phi(\lambda x) = |\lambda|\phi(x)$ for all $x \in X$ and all $\lambda \in \mathbb{R}$
(*ii*) $\phi(x) = 0$ if and only if $x = 0$

For every $\phi \in \Phi(X)$, we denote $S_\phi := \{x \in X : \phi(x) = 1\}$ and $C_\phi := \{x \in X : \phi(x) \leq 1\}$. Notice, that in general C_ϕ is not a convex set (resp. not closed), if we do not suppose that ϕ is a convex function (resp. a continuous function). It is easy to see that

$$(1) \quad C_\phi \text{ is bounded } \iff S_\phi \text{ is bounded } \iff \inf_{x \in X : \|x\|=1} \phi(x) > 0.$$

Thanks to the symmetry of $\phi \in \Phi(X)$ and the fact that $B(C_\phi) = B(S_\phi)$, we have using the notation of Theorem 1, that

$$B(C_\phi) = \mathcal{N}_\phi(X^*). \tag{2}$$

The polar of S_ϕ coincides with $\mathcal{N}_{\phi,1}(X^*)$ and we have

$$C_\phi^\circ \subset S_\phi^\circ = \mathcal{N}_{\phi,1}(X^*). \tag{3}$$

Now, using (1), (2) and (3) and applying Theorem 1 to the spaces $F = X^*$ and $Y = \mathbb{R}$ (using Exemple 1), we get directly the following informations about the size of the barrier cone, as well as the polar of a set of the form C_ϕ in the dual space.

Corollary 4. *Let X be a normed space and $\phi \in \Phi(X)$. If C_ϕ is not bounded in X, then the polar C_ϕ° is contained in a porous subset of X^*. Moreover, the followin assertions are equivalent.*
(i) $B(C_\phi) \neq X^*$.
(ii) C_ϕ is not bounded in X.
(iii) $B(C_\phi)$ is a σ-porous subset of X^*.

We deduce that, if K is any nonempty subset of X such that $S_\phi \subset K$, for some $\phi \in \Phi(X)$ with S_ϕ not bounded, then the polar K° is contained in a porous subset of X^* and the barrier cone $B(K)$ is contained in a σ-porous subset of X^*. Notice that if K is a closed absorbing disk in X, does not contain a non-trivial vector subspace and is a neighborhood of the origin in X, then the Minkowski functional ϕ_K of K is a continuous norm (with respect to the norm $\|\cdot\|$, but not equivalent to it, if K is not bounded) hence $\phi_K \in \Phi(X)$ and we have that $K = C_{\phi_K}$.

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