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Mohammed Bachir

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POROSITY IN THE SPACE OF HÖLDER-FUNCTIONS.

MOHAMMED BACHIR

ABSTRACT. Let \((X, d)\) be a bounded metric space with a base point \(0_X\), \((Y, \|\cdot\|)\) be a Banach space and \(\text{Lip}_0^\alpha(X, Y)\) be the space of all \(\alpha\)-Hölder-functions that vanish at \(0_X\), equipped with its natural norm \((0 < \alpha \leq 1)\). Let \(0 < \alpha < \beta \leq 1\). We prove that \(\text{Lip}_0^\beta(X, Y)\) is a \(\sigma\)-porous subset of \(\text{Lip}_0^\alpha(X, Y)\), if (and only if) \(\inf\{d(x, x') : x, x' \in X; x \neq x'\} = 0\) (i.e. \(d\) is non-uniformly discrete). A more general result will be given.

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1. Introduction

The main result of this note is Theorem 1, which gives a condition for some class of subsets of Lipschitz functions to be \(\sigma\)-porous subsets. The result in the abstract, as well as all the other results of this note, are just a very immediate consequence of this main result. However, the main motivations which led to the main theorem of this note, was precisely the result mentioned in the abstract.

Given a metric space \((X, d)\) with a distinguished point \(0_X\) (called a base point of \(X\)) and a Banach space \((Y, \|\cdot\|)\), we denote by \(\text{Lip}_0(X, Y)\) (or by \(\text{Lip}_0(X)\), if no ambiguity arises) the Banach space of all Lipschitz functions from \(X\) into \(Y\) that vanish at the base point \(0_X\), equipped with its natural norm defined by

\[
\|f\|_L := \sup\{|f(x) - f(x')| \over d(x, x') : x, x' \in X; x \neq x\}, \forall f \in \text{Lip}_0(X, Y).
\]

We denote simply \(\text{Lip}_0(X)\) or \(\text{Lip}_0(Y)\), if \(Y = \mathbb{R}\). The space \(L(X, Y)\) denotes the space of all linear bounded operators from \(X\) into \(Y\). The space \(X^*\) denotes the topological dual of \(X\). Notice that the space \(\text{Lip}_0(X, Y)\) can be isometrically identified to \(L(F(X), Y)\) where \(F(X)\) is the free-Lipschitz space over \(X\) introduced by Godefroy-Kalton in [2]. Let us recall the definition of \(\sigma\)-porosity.

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Definition 1. Let \((F,d)\) be a metric space and \(A\) be a subset of \(F\). A set \(A\) of \(F\) is called porous if there is a \(c \in (0,1)\) so that for every \(x \in A\) there are \((y_n) \subset F\) with \(y_n \to x\) and so that \(B(y_n,cd(y_n,x)) \cap A = \emptyset\) for every \(n\) (We denote by \(B(z,r)\) the closed ball with center \(z\) and radius \(r\)). A set \(A\) is called \(\sigma\)-porous if it can be represented as a union \(A = \bigcup_{n=0}^{+\infty} A_n\) of countably many porous sets (the porosity constant \(c_n\) may vary with \(n\)).

Every \(\sigma\)-porous set is of first Baire category. Moreover, in \(\mathbb{R}^n\), every \(\sigma\)-porous set is of Lebesgue measure zero. However, there does exist a non-\(\sigma\)-porous subset of \(\mathbb{R}^n\) which is of the first category and of Lebesgue measure zero (for more informations about \(\sigma\)-porosity, we refer to [8] and [6]).

The property \((P)\). Let \((X,d)\) be a metric space and \(Y\) be a Banach space. Let \(F\) be a nonempty (closed) convex cone of \(\text{Lip}_0(X,Y)\). We say that \(F\) satisfies property \((P)\) if there exists a positive constant \(K_F > 0\) depending only on \(F\) such that:

\[
(P) \quad \forall (x,x') \in X \times X, \exists p \in F : \|p\|_L \leq K_F \text{ and } \|p(x) - p(x')\| = d(x,x').
\]

This property is related to the Hahn-Banach theorem and norming sets.

Examples 1. The property \((P)\) satisfied in the following cases:

(i) If \(X\) is a normed space and \(F\) contains the space \(X^*e := \{x \mapsto p(x).e : p \in X^*\}\), where \(e \in Y\) is a fixed point such that \(\|e\| = 1\).

(ii) If \((X,d)\) is a metric space and \(F\) contains the functions \(d_z : x \mapsto d(x,z).e\), for all \(z \in X\), where \(e \in Y\) is a fixed point is such that \(\|e\| = 1\).

(iii) In particular, the space \(\text{Lip}_0(X,Y)\) satisfies the property \((P)\). If moreover, \(X\) is a normed space, then \(L(X,Y)\) has the property \((P)\) too.

Proof. (i) By the Hahn-Banach theorem, for all \(x \in X\) there exists \(x^* \in X^*\) such that \(\|x^*\| = 1\) and \(x^*(x) = \|x\|\). Then, for each \(x \in X\), we consider the continuous linear map \(p_x = x^*e : X \to Y\) defined by \(p_x(z) = x^*(z)e\) for all \(z \in X\), and the property \((P)\) is satisfied.

(ii) Immediate.

(iii) This part follows from (i) and (ii) respectively.

\[\square\]

2. The main result

We are going to give the proof of the main result of this note. Let \((X,d)\) be a metric space with a base point \(0\) and \(Y\) be a Banach space. Let \(F \subset \text{Lip}_0(X,Y)\) and \(\phi : X \times X \to \mathbb{R}^+\) be a positive function such that \(\phi(x,x') = 0\) if and only if \(x = x'\). For each real number \(s > 0\), we denote:

\[
\mathcal{N}_{\phi,s}(F) := \{f \in F : \sup_{x,x' \in X; x \neq x'} \frac{\|f(x) - f(x')\|}{\phi(x,x')} \leq s\},
\]

\[
\mathcal{N}_{\phi}(F) := \{f \in F : \sup_{x,x' \in X; x \neq x'} \frac{\|f(x) - f(x')\|}{\phi(x,x')} < +\infty\}.
\]
Notice that \( \mathcal{N}_\phi(F) = \bigcup_{k \in \mathbb{N}} \mathcal{N}_{\phi,k}(F) \) and \( \mathcal{N}_\psi(F) \subseteq \mathcal{N}_\phi(F) \) if \( \psi \leq \phi \).

**Theorem 1.** Let \( F \) be a nonempty (closed) convex cone of \( \text{Lip}_0(X,Y) \) satisfying \((\mathcal{P})\). Let \( \phi : X \times X \to \mathbb{R}^+ \) be any positive function such that \( \phi(x,x') = 0 \) if and only if \( x = x' \). Suppose that \( \inf \{ \frac{\phi(x,x')}{d(x,x')} : x, x' \in X; x \neq x' \} = 0 \), then for every positive real number \( s > 0 \), we have that \( \mathcal{N}_{\phi,s}(F) \) is a porous subset of \( (F,\| \cdot \|_L) \). Consequently, the following assertions are equivalent.

1. \( \mathcal{N}_\phi(F) \neq F \).
2. \( \inf \{ \frac{\phi(x,x')}{d(x,x')} : x, x' \in X; x \neq x' \} = 0 \).
3. \( \mathcal{N}_\phi(F) \) is a \( \sigma \)-porous subset of \( (F,\| \cdot \|_L) \).

**Proof.** (1) \( \implies \) (2). Suppose that \( \alpha := \inf \{ \frac{\phi(x,x')}{d(x,x')} : x, x' \in X; x \neq x' \} > 0 \), then \( \phi(x,x') \geq \alpha d(x,x') \) for all \( x,x' \in X \). It follows that for every \( f \in F \), we have that

\[
\sup_{x,x' \in X; x \neq x'} \frac{\| f(x) - f(x') \|}{\phi(x,x')} \leq \| f \|_L \sup_{x,x' \in X; x \neq x'} \frac{d(x,x')}{\phi(x,x')} \leq \| f \|_L \frac{\alpha}{\alpha} < +\infty.
\]

Thus, \( \mathcal{N}_\phi(F) = F \). Part (3) \( \implies \) (1) is trivial.

Let us prove that if \( \inf \{ \frac{\phi(x,x')}{d(x,x')} : x, x' \in X; x \neq x' \} = 0 \), then for every \( s > 0 \), we have that \( \mathcal{N}_{\phi,s}(F) \) is a porous subset of \( (F,\| \cdot \|_L) \), this gives in particular (2) \( \implies \) (3). Indeed, if \( \inf \{ \frac{\phi(x,x')}{d(x,x')} : x, x' \in X; x \neq x' \} = 0 \), then there exists a pair of sequences \( (a_n),(b_n) \subset X \) such that \( 0 < r_n := \frac{\phi(a_n,b_n)}{d(a_n,b_n)} \to 0 \). By assumption, there exists \( K_F > 0 \) and a sequence \( (p_n) \subset F \) such that \( \| p_n \|_L \leq K_F \) and \( \| p_n(a_n) - p_n(b_n) \| = d(a_n,b_n) \), for all \( n \in \mathbb{N} \). Let \( f \in \mathcal{N}_{\phi,s}(F) \), then we have that \( \sup_{x,x' \in X; x \neq x'} \frac{\| f(x) - f(x') \|}{\phi(x,x')} \leq s \). It follows that

\[
\frac{\| (f + \sqrt{r_n} p_n)(a_n) - (f + \sqrt{r_n} p_n)(b_n) \|}{\phi(a_n,b_n)} \geq \sqrt{r_n} \frac{\| p_n(a_n) - p_n(b_n) \|}{\phi(a_n,b_n)} - \| (f(a_n) - f(b_n)) \| \frac{\phi(a_n,b_n)}{\phi(a_n,b_n)} \\
\geq \sqrt{r_n} \frac{d(a_n,b_n)}{\phi(a_n,b_n)} - \sup_{x,x' \in X; x \neq x'} \frac{\| f(x) - f(x') \|}{\phi(x,x')} \\
\geq \frac{1}{\sqrt{r_n}} \to s
\]

Since, \( r_n \to 0 \), when \( n \to +\infty \), there exists a subsequence \( (r_m) \) such that

\[
\frac{1}{\sqrt{r_m}} > 4s, \quad \forall m \in \mathbb{N}.
\]

We set \( f_m = f + \sqrt{r_m} p_{m} \in F \), for all \( m \in \mathbb{N} \). We have that

\[
\| f_m - f \|_L = \sqrt{r_m} \| p_{m} \|_L \leq K_F \sqrt{r_m} \to 0 \text{ when } m \to +\infty.
\]
Let us prove that $B(f_m, \frac{1}{2K_F} \| f_m - f \|_L) \subset F \backslash N_{\phi,s}(F)$ for all $m \in \mathbb{N}$. Indeed, let $g \in B(f_m, \frac{1}{2} \| f_m - f \|_L)$, then we have using the above informations that
\[
\frac{\| g(a_{m,n}) - g(b_{m,n}) \|}{\phi(a_{m,n}, b_{m,n})} \geq \frac{\| f_m(a_{m,n}) - f_m(b_{m,n}) \|}{\phi(a_{m,n}, b_{m,n})} - \frac{\| (f_m - g)(a_{m,n}) - (f_m - g)(b_{m,n}) \|}{\phi(a_{m,n}, b_{m,n})}
\geq \left( \frac{1}{\sqrt{T_n}} - s \right) - \frac{1}{2K_F} \| f_m - f \|_L \frac{d(a_{m,n}, b_{m,n})}{\phi(a_{m,n}, b_{m,n})}
\geq \left( \frac{1}{\sqrt{T_n}} - s \right) - \frac{1}{2} \frac{1}{T_n} \frac{d(a_{m,n}, b_{m,n})}{\phi(a_{m,n}, b_{m,n})}
\geq \left( \frac{1}{\sqrt{T_n}} - s \right) - \frac{1}{2} \frac{1}{T_n} \frac{1}{T_n}
= \frac{1}{2} \frac{1}{T_n} - s
> s.
\]
Thus, we have that $g \in F \backslash N_{\phi,s}(F)$ and so that $B(f_m, \frac{1}{2} \| f_m - f \|_L) \subset F \backslash N_{\phi,s}(F)$ for all $m \in \mathbb{N}$. Thus, $N_{\phi,s}(F)$ is porous in $F$ (with $c = \frac{1}{2K_F}$). It follows that $N_{\phi}(F) = \bigcup_{k \in \mathbb{N}} N_{\phi,k}(F)$ is $\sigma$-porous in $(F, \| \cdot \|_L)$. \hfill \Box

2.1 Immediate consequences. We deduce immediately the result mentioned in the abstract.

**Corollary 1.** Let $X_1 := (X,d_1)$ and $X_2 := (X,d_2)$ be a set equipped with two metrics such that $d_1 \leq d_2$ and let $(Y,\| \cdot \|)$ be a Banach space. Then, $\text{Lip}_0(X_1,Y)$ is a $\sigma$-porous subset of $\text{Lip}_0(X_2,Y)$ if (and only if) $d_1$ and $d_2$ are not equivalent, if and only if $\text{Lip}_0(X_1,Y) \neq \text{Lip}_0(X_2,Y)$.

**Proof.** We use Theorem 1 and part (iii) of Exemple 1 observing the following equality $\text{Lip}_0(X_1,Y) = N_{d_1}(\text{Lip}_0(X_2,Y))$. \hfill \Box

Notice that if $0 < \alpha \leq 1$ and $d$ is a metric, so is $d^\alpha$, hence the above corollary applies to the space of $\alpha$-Hölder-functions that vanish at $0_X$ which is $\text{Lip}_0^\alpha(X,Y) := \text{Lip}_0(X,d^\alpha,Y)$. Notice also that if $0 < \alpha < \beta \leq 1$ and $d$ is bounded, then $\text{Lip}_0^\alpha(X,Y) \subset \text{Lip}_0^\beta(X,Y)$. The metrics $d^\alpha$ and $d^\beta$ are not equivalent if and only if $\inf \{ d(x,x') : x,x' \in X; x \neq x' \} = 0$, if and only if $\inf \{ d(x,x') : x,x' \in X; x \neq x' \} = 0$ (since $\beta > \alpha$). Thus, we get the result of the abstract.

**Corollary 2.** Let $(X,d)$ be a bounded metric space with a base point $0_X$, $(Y,\| \cdot \|)$ be a Banach space and $0 < \alpha < \beta \leq 1$. Then, $\text{Lip}_0^\alpha(X,Y)$ is a $\sigma$-porous subset of $\text{Lip}_0^\beta(X,Y)$, if and only if $\inf \{ d(x,x') : x,x' \in X; x \neq x' \} = 0$.

Similarly to the case of lipschitz spaces, we obtain the following results in the linear case.
Proposition 1.

Theorem 2.11). Then the image of the adjoint $i^{\ast}$ of the adjoint $i$ of $T$ of a one-to-one bounded linear operator $T$ and the non-$\sigma$-porosity of its image (see in this spirit, the open mapping theorem in [7, Theorem 2.11]).

Example 1. Let $i : (l^1(\mathbb{N}), \| \cdot \|_1) \to (l^1(\mathbb{N}), \| \cdot \|_\infty)$ be the continuous identity map. Then the image of the adjoint $i^{\ast}$ of $i$ is a $\sigma$-porous subset of $(l^\infty(\mathbb{N}), \| \cdot \|_\infty)$.

We give in the following corollary a connexion between the surjectivity of the adjoint $T^{\ast}$ of a one-to-one bounded linear operator $T$ and the non-$\sigma$-porosity of its image (see in this spirit, the open mapping theorem in [7, Theorem 2.11]).

Proposition 1. Let $(X, \| \cdot \|_X)$ and $(Z, \| \cdot \|_Z)$ be Banach spaces. Let $T : X \to Z$ be a one-to-one bounded linear operator and $T^{\ast}$ its adjoint. Then, the following assertions are equivalent.

(i) $T^{\ast}(Z^{\ast})$ is not a $\sigma$-porous subset of $X^{\ast}$.

(ii) There exists $\alpha > 0$ such that $\alpha \|x\|_X \leq \|T(x)\|_Z$ for all $x \in X$.

(iii) $T^{\ast}$ is onto.

Proof. Since $T : X \to Z$ is a one-to-one bounded linear operator, then, the following map define another norm on $X$:

$$\|x\| := \frac{\|T(x)\|_Z}{\|T\|} \leq \|x\|_X, \; \forall x \in X.$$ 

Let us denote $X_1 := (X, \| \cdot \|)$. By Corollary 3, applied with $Y = \mathbb{R}$, we have that $X_1^{\ast}$ is a $\sigma$-porous subset of $X^{\ast}$ if and only if $\| \cdot \|$ and $\| \cdot \|_X$ are not equivalent. Thus, if (ii) is not satisfied (that is, $\| \cdot \|$ and $\| \cdot \|_X$ are not equivalent) then, since $T^{\ast}(Z^{\ast}) \subset X_1^{\ast}$ we get that $T^{\ast}(Z^{\ast})$ is contained in a $\sigma$-porous subset of $X^{\ast}$. Hence, (i) $\implies$ (ii) is proved. Now, suppose that (ii) holds, it follows that $T(X)$ is closed in $Y$. Let $x^{\ast} \in X^{\ast}$ and define $\phi$ on $T(X)$ by $\phi(T(x)) := x^{\ast}(x)$ for all $x \in X$. Clearly $\phi$ is well defined (since $T$ is one-to-one) and linear continuous on $T(X)$. Thus, $\phi$ extends to a linear continuous functional $y^{\ast} \in Y^{\ast}$ and we have $T^{\ast}(y^{\ast}) = y^{\ast} \circ T = x^{\ast}$. Hence, $T^{\ast}$ is onto and (ii) $\implies$ (iii) is proved. Part (iii) $\implies$ (i), is trivial.

Let $(X, \| \cdot \|)$ be a normed space, and let $S$ be a nonempty subset of the dual space $X^{\ast}$. The set $S$ is called separating if: $x^{\ast}(x) = 0$ for all $x^{\ast} \in S$ implies that $x = 0$. It is called norming if the functional $N_S(x) = \sup_{x^{\ast} \in S; x^{\ast} \neq 0} \frac{|x^{\ast}(x)|}{\|x^{\ast}\|}$, is an equivalent norm on $X$ (see [2] for the use of this notion).
Proposition 2. Let \((X, \| \cdot \|)\) be a normed space. Every separating subset 
\(S \subset X^*\) which is not a \(\sigma\)-porous subset of \(X^*\), is norming.

Proof. It is clear that \(N(x) \leq \|x\|\) for all \(x \in X\). On the other hand, we have that 
\[ S \subset N_{S}(X) := \{ x^* \in X^* : \sup_{N_{S}(x) = 1} |x^*(x)| < +\infty \} \]
Since \(S\) is not contained in a \(\sigma\)-porous subset of \(X^*\), then \(N_{\phi}(X)\) must be non-\(\sigma\)-porous, which implies from Theorem 1 that \(\inf_{\|x\|=1} N_{S}(x) > 0\). Hence \(N_{S}\) is equivalent to \(\| \cdot \|\). \(\Box\)

2.2. Coarse Lipschitz function and Lipschitz-free space. Given a metric space \((X, d)\) with a base point \(0_X\), the free space \(F(X)\) is constructed as follows: we first consider as pivot space the Banach space \((\text{Lip}_0(X), \| \cdot \|_{L})\) of real-valued Lipschitz functions vanishing at the base point. Then each \(x \in X\) is identified to a Dirac measure \(\delta_x\) acting linearly on \(\text{Lip}_0(X)\) as evaluation. Then the mapping 
\[ \delta_X : X \to \text{Lip}_0(X)^* \]
that maps \(x\) to \(\delta_x\) is an isometric embedding. The Lipschitz-free space \(F(X)\) over \(X\) is defined as the closed linear span of \(\delta(X)\) in \(\text{Lip}_0(X)\). Furthermore, the free space is a predual for \(\text{Lip}_0(X)\), meaning that \(F(X)^*\) is isometrically isomorphic to \(\text{Lip}_0(X)\). Let \((X, d)\) and \((Y, d')\) be two metric spaces, each one with a base point \((0_X, 0_Y)\), respectively and \(F : X \to Y\) a Lipschitz function such that \(R(0_X) = 0_Y\). Then, it is well known (see [2, Lemma 2.2]) that there exists a unique linear operator \(\tilde{F} : F(X) \to F(Y)\) such that \(\|F\|_{L} = \|\tilde{F}\|\) and \(\delta_Y \circ F = \tilde{F} \circ \delta_X\). The adjoint of \(\tilde{F}\), namely \(\tilde{F}^* : \text{Lip}_0(Y) \to \text{Lip}_0(X)\), satisfies \(\tilde{F}^*(f) = f \circ F\) for all \(f \in \text{Lip}_0(Y)\).

A map \(F : (X, d) \to (Y, d')\) is said to be a coarse Lipschitz, if there exist \(\alpha, \beta > 0\) such that 
\[ \alpha d(x, x') \leq d'(F(x), F(x')) \leq \beta d(x, x'), \quad \forall x, x' \in X. \]
Combining Proposition 1 together with a similar proof, we obtain in the following proposition, a characterization of coarse Lipschitz maps.

Proposition 3. Let \((X, d)\) and \((Y, d')\) be metric spaces with base points \(0_X\) and \(0_Y\) respectively and let \(F : (X, d) \to (Y, d')\) be a one-to-one Lipschitz map such that \(F(0_X) = 0_Y\). Then the following assertions are equivalent.
(i) The image of \(\tilde{F}^*\) is not \(\sigma\)-porous in \(\text{Lip}_0(X)\).
(ii) The map \(F\) is coarse Lipschitz.
(iii) The adjoint \(\tilde{F}^*\) is onto.
(iv) The linear map \(\tilde{F} : F(X) \to F(Y)\) is coarse Lipschitz.
Proof. Since, $F$ is one-to-one, we define the following metric on $X$

$$d_1(x, x') := \frac{1}{L_F}d'(F(x), F(x')) \leq d(x, x'), \quad \forall x, x' \in X,$$

where $L_F$ denotes the constant of Lipschitz of $F$. Suppose that $F$ is not coarse Lipschitz, then the metric $d_1$ is not equivalent to the metric $d$. It follows, using Corollary 1, that Lip$_0(X_1)$ is $\sigma$-porous subset of Lip$_0(X)$, where $X_1 = (X, d_1)$.

Now, we observe that $\text{Im}(\hat{F}^*) := \{f \circ F : f \in \text{Lip}_0(Y)\} \subset \text{Lip}_0(X_1)$, which implies that $\text{Im}(\hat{F}^*)$ is a $\sigma$-porous subset of $(\text{Lip}_0(X), \|\cdot\|_L)$. Thus, we proved that (i) $\implies$ (ii). Let us prove that (ii) $\implies$ (iii). Let $g \in \text{Lip}_0(X)$, we need to show that there exists $f \in \text{Lip}_0(Y)$ such that $g = f \circ F$. Indeed, define $\phi : F(X) \to \mathbb{R}$ by $\phi(F(x)) := g(x)$ for all $x \in X$. The map $\phi$ is well defined since $F$ is one-to-one. On the other hand, $\phi$ is Lipschitz on $F(X)$ since $F$ is coarse Lipschitz. Thus, $\phi$ extends to a Lipschitz function $f$ from $Y$ into $\mathbb{R}$ with the same constant of Lipschitz, by the inf-convolution formula ($L_\phi$ denotes the constant of Lipschitz of $\phi$ on $F(X)$): $\forall y \in Y$

$$f(y) := \inf\{\phi(y') + L_\phi d(y, y') : y' \in F(X)\}.$$ 

Hence, $f \in \text{Lip}_0(Y)$ and $f \circ F(x) = \phi(F(x)) = g(x)$ for all $x \in X$ and so (ii) $\implies$ (iii) is proved. Part (iii) $\implies$ (i) is trivial. Now, from Proposition 1, we see (iii) $\iff$ (iv).

2.3. Application to the barrier cone and polar of sets. Let $X$ be a normed space and $K$ be a nonempty subset of $X$. The barrier cone of $K$ is the subset $B(K)$ of the topological dual $X^*$ defined by

$$B(K) = \{x^* \in X^* : \sup_{x \in K} x^*(x) < +\infty\}.$$ 

The polar set of $K$ is a subset of the barrier cone of $K$ defined as follows:

$$K^o = \{x^* \in X^* : \sup_{x \in K} x^*(x) \leq 1\}.$$ 

The study of barrier cones has interested several authors. It is shown in [5, Theorem 3.1.1] that for a closed convex subset $K$ of $X$, we have that $B(K) = X^*$ if and only if $K$ is bounded, on the other hand, $B(K)$ is dense in $X^*$ if and only if $K$ does not contain any halfline. In general, we know that $B(K) \neq X^*$ (see example in [1]). A study of the closure of the barrier of a closed convex set is given in [1]. As an immediate consequence of main theorem, we obtain below that the barrier cone of some general class of unbounded subsets is a $\sigma$-porous subset of the dual $X^*$. This shows that in general, the barrier cone may be a "very small" subset of $X^*$.

We define the classe $\Phi(X)$ of positive functions on $X$ (not necessarily continuous) as follows: $\phi \in \Phi(X)$ if and only if, $\phi : X \to \mathbb{R}^+$ and satisfies
(i) \( \phi(\lambda x) = |\lambda| \phi(x) \) for all \( x \in X \) and all \( \lambda \in \mathbb{R} \)

(ii) \( \phi(x) = 0 \) if and only if \( x = 0 \)

For every \( \phi \in \Phi(X) \), we denote \( S_\phi := \{ x \in X : \phi(x) = 1 \} \) and \( C_\phi := \{ x \in X : \phi(x) \leq 1 \} \). Notice, that in general \( C_\phi \) is not a convex set (resp. not closed), if we do not suppose that \( \phi \) is a convex function (resp. a continuous function). It is easy to see that

\( C_\phi \) is bounded \( \iff \) \( S_\phi \) is bounded \( \iff \) \( \inf_{x \in X : \| x \| = 1} \phi(x) > 0 \).

Thanks to the symmetry of \( \phi \in \Phi(X) \) and the fact that \( B(C_\phi) = B(S_\phi) \), we have using the notation of Theorem 1, that

\( B(C_\phi) = N_\phi(X^*) \).

The polar of \( S_\phi \) coincides with \( N_{\phi,1}(X^*) \) and we have

\( C_\phi^\circ \subset S_\phi^* = N_{\phi,1}(X^*) \).

Now, using (1), (2) and (3) and applying Theorem 1 to the spaces \( F = X^* \) and \( Y = \mathbb{R} \) (using Exemple 1), we get directly the following informations about the size of the barrier cone, as well as the polar of a set of the form \( C_\phi \) in the dual space.

**Corollary 4.** Let \( X \) be a normed space and \( \phi \in \Phi(X) \). If \( C_\phi \) is not bounded in \( X \), then the polar \( C_\phi^\circ \) is contained in a porous subset of \( X^* \). Moreover, the followin assertions are equivalent.

(i) \( B(C_\phi) \neq X^* \).

(ii) \( C_\phi \) is not bounded in \( X \).

(iii) \( B(C_\phi) \) is a \( \sigma \)-porous subset of \( X^* \).

We deduce that, if \( K \) is any nonempty subset of \( X \) such that \( S_\phi \subset K \), for some \( \phi \in \Phi(X) \) with \( S_\phi \) not bounded, then the polar \( K^\circ \) is contained in a porous subset of \( X^* \) and the barrier cone \( B(K) \) is contained in a \( \sigma \)-porous subset of \( X^* \). Notice that if \( K \) is a closed absorbing disk in \( X \), does not contain a non-trivial vector subspace and is a neighborhood of the origin in \( X \), then the Minkowski functional \( \phi_K \) of \( K \) is a continuous norm (with respect to the norm \( \| \cdot \| \), but not equivalent to it, if \( K \) is not bounded) hence \( \phi_K \in \Phi(X) \) and we have that \( K = C_{\phi_K} \).

**References**


Laboratoire SAMM 4543, Université Paris 1 Panthéon-Sorbonne, Centre P.M.F. 90 rue Tolbiac, 75634 Paris cedex 13, France
Email address: Mohammed.Bachir@univ-paris1.fr