Metrization of probabilistic metric spaces. Applications to fixed point theory and Arzela-Ascoli type theorem

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METRIZATION OF PROBABILISTIC METRIC SPACES. APPLICATIONS TO FIXED POINT THEORY AND ARZELA-ASCOLI TYPE THEOREM

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Abstract. Schweizer, Sklar and Thorp proved in 1960 that a Menger space (G, D, T) under a continuous t-norm T, induce a natural topology τ wich is metrizable. We extend this result to any probabilistic metric space $(G, D, \star)$ provided that the triangle function $\star$ is continuous. We prove in this case, that the topological space (G, τ) is uniformly homeomorphic to a (deterministic) metric space (G, σ_D) for some canonical metric σ_D on G. As applications, we extend the fixed point theorem of Hicks to probabilistic metric spaces which are not necessarily Menger spaces and we prove a probabilistic Arzela-Ascoli type theorem.

Keywords: Metrization of probabilistic metric space; Probabilistic 1-Lipschitz map; Probabilistic Arzela-Ascoli type Theorem; Probabilistic fixed point theorem.

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1. Introduction

Let (G, D, T) be a Menger space equipped with a probabilistic metric D and a t-norm T (the definitions and notation reminders will be given in the details in Section 2). Schweizer and Sklsar [13] defined for $\varepsilon, \lambda > 0$ and each $x \in G$ a neighborhood $N_x(\varepsilon, \lambda)$ as follows

$$N_x(\varepsilon, \lambda) = \{y \in G : D(x, y)(\varepsilon) > 1 - \lambda\}.$$

Schweizer, Sklar and Thorp proved in [14] that, given a t-norm T of a Menger space (G, D, T) satisfying $1 = \sup_{x<1} T(x, x)$ (in particular if T is continuous), the collection $\{N_x(\varepsilon, \lambda) : x \in G\}$ taken as a neighborhood base at x gives rise to a metrizable topology. In [11] Morrel and Nagata proved the following two extensions:

(1) The class of topological Menger spaces coincides with that of semi-metrizable topological spaces.
(2) No condition on T weaker than $1 = \sup_{x<1} T(x, x)$ can guarantee that a Menger space, under T, is topological.

The aim of the present paper is to prove that, in a general probabilistic metric space $(G, D, \star)$, not necessarily being a Menger space, the collection $\{N_x(\varepsilon, \lambda) : x \in X\}$ taken as a neighborhood base at x gives rise to a topology which is uniformly homeomorphic to a metric space, provided that the triangle function $\star$ is continuous (necessarily uniformly continuous by Sibley's result in [16] on the compactness of

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$(\Delta^+, d_{\mathcal{L}})$, where $d_{\mathcal{L}}$ denotes the modified Lévy distance and Δ^+ denotes the set of all nondecreasing and left-continuous distributions that vanish at 0).

We get an even more precise result : if $w_\star : [0, +\infty] \to [0, +\infty]$ is a modulus of uniform continuity for the triangle function $\star$, then the (deterministic) metric σ_D on G defined canonically from the probabilistic metric D by

$$\forall x, y \in G, \quad \sigma_D(x, y) := \sup_{z \in G} d_{\mathcal{L}}(D(x, z), D(z, y)),$$

satisfies the following inequalities

$$\forall x, y \in G, \; d_{\mathcal{L}}(D(x, y), \mathcal{H}_0) \leq \sigma_D(x, y) \leq w_\star(d_{\mathcal{L}}(D(x, y), \mathcal{H}_0)). \tag{1}$$

Note from [12] that $y \in N_x(t, t)$ if and only if $d_{\mathcal{L}}(D(x, y), \mathcal{H}_0) < t$, for all $t > 0$. If moreover we assume that $\star$ is k-Lipschitz given some positive real number k (necessarily $k \geq 1$), then we can take $w_\star(t) = kt$ for all $t \geq 0$ (see Proposition 4 for examples of such functions). As an immediate consequence of (1), the semi-metric $\alpha(x, y) := d_{\mathcal{L}}(D(x, y), \mathcal{H}_0)$ define a topology on $(G, D, \star)$ which is uniformly (resp. Lipschitz) homeomorphic to the metric space (G, σ_D), whenever $\star$ is continuous (resp. Lipschitz continuous). This result is an extension to non necessarily Menger spaces of the works established in Menger spaces by Schweizer, Sklar and Thorp in [14]. In particular, the formula (1) allows us to transfer several known results from metric space theory to the probabilistic metric theory. For instance, using (1) and the Ekeland variational principle we give some extensions of the fixed point theorem of Hicks (see [4]), or using again (1) we give an Arzela-Ascoli type theorem for the the space of probabilistic 1-Lipschtz maps introduced recently in [1]. Notice that other results such as Baire theorem and all its variants/consequences can be transfered, thanks to our result, to the probabilistic metric framework.

This paper is organized as follows. In Section 2, we recall some classical notions related to probabilistic metric space. In Section 3, we treat the metrization of probabilistic metric space and prove Theorem 1. We also give some new properties. In Section 4, we establish fixed point theorems (Theorem 2 and Theorem 3) extending a result of Hicks (see [4]). In Section 5, we prove Theorem 5, showing that the set of probabilistic 1-Lipschitz maps introduced in [1] is a compact space for the uniform convergence, giving a probabilistic Arzela-Ascoli theorem.

2. Definitions and notation

In this section, we recall some known facts about probabilistic metric spaces, the modified Lévy distance and the weak convergence. All these notions can be found in [12], [5] and [6]. We also recall the notion of probabilistic 1-Lipschitz map introduced in [1], which shall play an important role in the sequel.

2.1. **Probabilistic metric space and triangle function.** By Δ^+ we denote the set of all (cumulative) distribution functions $F : [-\infty, +\infty] \longrightarrow [0, 1]$, nondecreasing and left-continuous with $F(-\infty) = 0$; $F(+\infty) = 1$ and $F(0) = 0$. For $a \in [0, +\infty[$, we denote $\mathcal{H}_a(t) = 0$ if $t \leq a$ and $\mathcal{H}_a(t) = 1$, if $t > a$.

In the sequel, we shall write $F \leq G$ for

$$\forall t \in \mathbb{R}, \quad F(t) \leq G(t),$$

which defines an ordering relation on Δ^+.

Definition 1. *([12, 4, 5, 6]) A binary operation $\star$ on Δ^+ is called a triangle function if and only if it is commutative, associative, non-decreasing in each place, and has $\mathcal{H}_0$ as neutral element. In other words:*

(i) $F \star L \in \Delta^+$ *for all* $F, L \in \Delta^+$.
(ii) $F \star L = L \star F$ *for all* $F, L \in \Delta^+$.
(iii) $F \star (L \star K) = (F \star L) \star K$, *for all* $F, L, K \in \Delta^+$.
(iv) $F \star \mathcal{H}_0 = F$ *for all* $F \in \Delta^+$.
(v) $F \leq L \Longrightarrow F \star K \leq L \star K$ *for all* $F, L, K \in \Delta^+$.

Definition 2. *A t-norm is a function* $T : [0,1] \times [0,1] \to [0,1]$, *usually called a triangular norm (see [12, 4, 5, 6]), satisfying*

- $T(x,y) = T(y,x)$ *(commutativity);*
- $T(x, T(y,z)) = T(T(x,y),z)$ *(associativity);*
- $T(x,y) \leq T(x,z)$ *whenever* $y \leq z$ *(monotonicity);*
- $T(x,1) = x$ *(boundary condition).*

Definition 3. *A probabilistic metric space* $(G, D, \star)$ *(an PM-space) is a set* G *together with a triangle function* $\star$ *and a function* $D : G \times G \to \Delta^+$ *satisfying:*

(i) $D(x,y) = \mathcal{H}_0$ *iff* $x = y$.
(ii) $D(x,y) = D(y,x)$ *for all* $x, y \in G$
(iii) $D(x,y) \star D(y,z) \leq D(x,z)$ *for all* $x, y, z \in G$

Usually, $D(x,y)$ is denoted by $F_{x,y}$ in the literature. A probabilistic metric space $(G, D, \star)$ is called a Menger space and denoted by (G, D, T), iff the triangle function $\star := \star_T$ is defined from a t-norm T as follows: for all $F, L \in \Delta^+$ and for all $t \in \mathbb{R}$,

$$\begin{aligned}(F \star_T L)(t) &:= \sup_{u+v=t} T(F(u), L(v)) \\ &= \sup_{u,v \leq 0 : u+v=t} T(F(u), L(v))\end{aligned} \tag{2}$$

2.2. **Lévy distance and weak convergence.**

Definition 4. *Let* F *and* G *be in* Δ^+. *For any* $h > 0$ *we set*

$$A^h_{F,G} = \{t \geq 0 \ st. \ G(t) \leq F(t+h) + h\}.$$

The modified Lévy distance is the map $d_{\mathcal{L}}$ *defined on* $\Delta^+ \times \Delta^+$ *as*

$$d_{\mathcal{L}}(F,G) = \inf\left\{h > 0 \ st. \ [0, h^{-1}[\subset A^h_{F,G} \cap A^h_{G,F}\right\}.$$

Notice that, for all F, $G \in \Delta^+$,

(i) if $F \leq G$ then $A_{G,F} = [0, +\infty[$, hence
$$d_{\mathcal{L}}(F,G) = \inf\left\{h > 0 \text{ st. } [0, h^{-1}[\subset A^h_{F,G}\right\}.$$
(ii) if $h \geq 1$, $A^h_{F,G} = A^h_{G,F} = [0, +\infty[$, hence $d_{\mathcal{L}}(F,G) \leq 1$.
(iii) The usual Levy distance between general cumulative distribution functions can be expressed as
$$\inf\left\{h > 0 \text{ st. } A^h_{F,G} = A^h_{G,F} = [0, +\infty[\right\}.$$
It is invariant under the action of translations which, as we shall see later, is not the case for the modified version since it somehow does not see the behaviour at infinity.

Definition 5. *Let $\star$ be a triangle function on Δ^+.*

(1) *A sequence (F_n) of distributions in Δ^+ converges weakly to a function F in Δ^+ if $(F_n(t))$ converges to $F(t)$ at each point t of continuity of F. In this case, we write indifferently $F_n \xrightarrow{w} F$ or $\lim_n F_n = F$.*

(2) *We say that the law $\star$ is continuous at $(F, L) \in \Delta^+ \times \Delta^+$ if we have $F_n \star L_n \xrightarrow{w} F \star L$, whenever $F_n \xrightarrow{w} F$ and $L_n \xrightarrow{w} L$.*

We recall the following results due to D. Sibley in [16, Theorem 1. and Theorem 2].

Lemma 1. *([16, 12]) The function $d_{\mathcal{L}}$ is a metric on Δ^+ and $(\Delta^+, d_{\mathcal{L}})$ is compact.*

Lemma 2. *([16, 12]) Let (F_n) be a sequence of functions in Δ^+, and let F be an element of Δ^+. Then (F_n) converges weakly to F if and only if $d_{\mathcal{L}}(F_n, F) \longrightarrow 0$, when $n \longrightarrow +\infty$.*

Remark 1. Thanks to Lemma 2, we shall indifferently use the notations $F_n \xrightarrow{w} F$ or $d_{\mathcal{L}}(F_n, F) \longrightarrow 0$ to say that (F_n) converges weakly to F.

Definition 6. *Let $(G, D, \star)$ be a probabilistic metric space. For $x \in G$ and $t > 0$, the strong t-neighborhood of x is the set*

$$N_x(t) = \{y \in G : D(x, y)(t) > 1 - t\},$$

and the strong neighborhood system for G is $\{N_x(t); x \in G, t > 0\}$.

Lemma 3. *([12, Lemme 4.3.3]) Let $t > 0$ and $x, y \in G$. Then we have $y \in N_x(t)$ if and only if $d_{\mathcal{L}}(D(x, y), \mathcal{H}_0) < t$.*

2.3. **Probabilistic 1-Lipschitz map.**

Definition 7. *Let $(G, D, \star)$ be a probabilistic metric space and let f be a function $f : (G, D, \star) \longrightarrow (\Delta^+, d_{\mathcal{L}})$. We say that f is a probabilistic 1-Lipschitz map if :*

$$\forall x, y \in G,\ D(x, y) \star f(y) \leq f(x).$$

We can also define probabilistic k-Lipschitz maps for any nonegative real number $k \geq 0$ as the maps f satisfying

$$\forall x, y \in G,\ D_k(x, y) \star f(y) \leq f(x),$$

where, for all $x, y \in G$ and all $t \in \mathbb{R}$, $D_k(x, y)(t) = D(x, y)(\frac{t}{k})$ if $k > 0$ and $D_0(x, y)(t) = \mathcal{H}_0(t)$ if $k = 0$. For sake of simplicity, when we use the notion in Definition 7, we shall only treat in this paper the case of probabilistic 1-Lipschitz maps, but our main result result could be easily extended to this more general setting.

Examples 1. Let (G, d) be a metric space. Assume that $\star$ is a triangle function on Δ^+ satisfying $\mathcal{H}_a \star \mathcal{H}_b = \mathcal{H}_{a+b}$ for all $a, b \in \mathbb{R}^+$ (for example if $\star = \star_T$ where T is a lef-continuous triangular norm). Let $(G, D, \star)$ be the probabilistic metric space defined with the probabilistic metric

$$D(p, q) = \mathcal{H}_{d(p,q)}.$$

Let $L : (G, d) \longrightarrow \mathbb{R}^+$ be a real-valued map. Then, L is a non-negative 1-Lipschitz map if and only if $f : (G, D, \star) \longrightarrow \Delta^+$ defined for all $x \in G$ by

$$f(x) := \mathcal{H}_{L(x)}$$

is a probabilistic 1-Lipschitz map. This example shows that the framework of probabilistic 1-Lipschitz maps encompasses the classical determinist case.

By $Lip^1_\star(G, \Delta^+)$ we denote the space of all probabilistic 1-Lipschitz maps

$$Lip^1_\star(G, \Delta^+) := \{f : G \longrightarrow \Delta^+ / D(x, y) \star f(y) \leq f(x); \forall x, y \in G\}.$$

For all $x \in G$, by δ_x we denote the map

$$\begin{array}{rcl} \delta_x : G & \longrightarrow & \Delta^+ \\ y & \mapsto & D(y, x). \end{array}$$

It follows from the properties of the probabilistic metric D that δ_x is a probabilistic 1-Lipschitz for every $x \in G$. We set $\mathcal{G}(G) := \{\delta_x, x \in G\}$ and by δ, we denote the operator

$$\begin{array}{rcl} \delta : G & \longrightarrow & \mathcal{G}(G) \subset Lip^1_\star(G, \Delta^+) \\ x & \mapsto & \delta_x. \end{array}$$

2.4. **Modulus of uniform continuity of a triangle function on Δ^+.** Let $\star : \Delta^+ \times \Delta^+ \to \Delta^+$ be a continuous triangle function (with respect to the modified Lévy distance $d_\mathcal{L}$). Since $(\Delta^+, d_\mathcal{L})$ is a compact metric space (see Lemma 1) and $\star$ is continuous, then $\star$ is uniformly continuous from $\Delta^+ \times \Delta^+$ into Δ^+. Let $\omega_\star : [0, +\infty] \to [0, +\infty]$ be a modulus of uniform continuity for $\star$ ($\lim_{t \to 0} \omega_\star(t) = \omega_\star(0) = 0$), that is for all $(F, L), (F', L') \in \Delta^+ \times \Delta^+$

$$d_\mathcal{L}(F \star L, F' \star L') \leq \omega_\star(d_\mathcal{L}(F, F') + d_\mathcal{L}(L, L')).$$

In particular for all $(F, L) \in \Delta^+ \times \Delta^+$

$$d_\mathcal{L}(F \star L, L) \leq \omega_\star(d_\mathcal{L}(F, \mathcal{H}_0)). \tag{3}$$

If moreover the operation $\star$ is k-Lipschitz (with respect to $d_\mathcal{L}$) for some positive number k then $\omega_\star(t) = kt$ for all $t \geq 0$ (necessarily $k \geq 1$, by using 3 with $L = \mathcal{H}_0$) is a modulus of uniform continuity. We give in Proposition 4 examples of k-Lipschitz triangle function using k-Lipschitz t-norms.

3. Metrization of Probabilistic Metric space.

We give below the main result of this section, that is a metrization of probabilistic metric space extending the result of Schweizer, Sklar and Thorp in [14].

Let $(G, D, \star)$ be a probabilistic metric space. We define canonically the metric σ_D on G using the probabilistic metric D as follows: for all $x, y \in G$

$$\sigma_D(x, y) := \sup_{z \in K} d_\mathcal{L}(D(x, z), D(y, z)) := \sup_{z \in K} d_\mathcal{L}(\delta_x(z), \delta_y(z)) := d_\infty(\delta_x, \delta_y)$$

It is easy to see that σ_D is a metric on G and that for all $x, y \in G$

$$d_\mathcal{L}(D(x, y), \mathcal{H}_0) \leq \sigma_D(x, y).$$

Theorem 1. *Let $(G, D, \star)$ be a probabilistic metric space such that $\star$ is continuous (resp. k-lipschitz). Let $\omega_\star$ be a modulus of uniform continuity of $\star$ on Δ^+. Then, the metric σ_D satisfies: for all $x, y \in G$*

$$d_\mathcal{L}(D(x, y), \mathcal{H}_0) \leq \sigma_D(x, y) \leq \omega_\star(d_\mathcal{L}(D(x, y), \mathcal{H}_0)).$$

$$(\text{resp. } d_\mathcal{L}(D(x, y), \mathcal{H}_0) \leq \sigma_D(x, y) \leq k d_\mathcal{L}(D(x, y), \mathcal{H}_0)).$$

In particular, the identity map $i : (G, \tau) \to (G, \sigma_D)$ is an uniform homeomorphism, where τ is the topology induced by the strong neighborhood system $\{N_x(t); x \in G, t > 0\}$ (see Definition 6).

This theorem is a mere consequence of the following lemma, that we will also use for proving Theorem 5.

Lemma 4. *Let $(G, D, \star)$ be a probabilistic metric space such that $\star$ is continuous. Let $\omega_\star$ be a modulus of uniform continuity of $\star$ on Δ^+. Then, the set $Lip^1_\star(G, \Delta^+)$ is uniformly equicontinuous. More precisely, we have $\forall x, y \in G$:*

$$\sup_{f \in Lip^1_\star(G,\Delta^+)} d_{\mathcal{L}}(f(x), f(y)) \leq \omega_\star(d_{\mathcal{L}}(D(x, y), \mathcal{H}_0)).$$

Proof. From the formula (3) about the modulus of uniform continuity of $\star$, we have that $\forall L \in \Delta^+, \forall x, y \in G$:

$$d_{\mathcal{L}}(D(x, y) \star L, L) \leq \omega_\star(d_{\mathcal{L}}(D(x, y), \mathcal{H}_0)).$$

In particular, we have for all $f \in Lip^1_\star(G, \Delta^+)$ and all x, $y \in G$,

$$\max[d_{\mathcal{L}}(D(x, y) \star f(x), f(x)), d_{\mathcal{L}}(D(x, y) \star f(y), f(y))] \leq \omega_\star(d_{\mathcal{L}}(D(x, y), \mathcal{H}_0)),$$

hence, it is enough to prove that

$$d_{\mathcal{L}}(f(x), f(y)) \leq \max[d_{\mathcal{L}}(D(x, y) \star f(x), f(x)), d_{\mathcal{L}}(D(x, y) \star f(y), f(y))].$$

Let $h_1, h_2 > 0$ such that,

$$[0, h_1^{-1}[\subset A^{h_1}_{D(x,y)\star f(x),f(x)} \cap A^{h_1}_{f(x),D(x,y)\star f(x)}. \tag{4}$$

$$[0, h_2^{-1}[\subset A^{h_2}_{D(x,y)\star f(y),f(y)} \cap A^{h_2}_{f(y),D(x,y)\star f(y)}. \tag{5}$$

that is, for all $t \in]0, h_1^{-1}[$ and all $t' \in]0, h_2^{-1}[$, we have

$$\begin{aligned} 0 \leq D(x, y) \star f(x)(t) &\leq f(x)(t + h_1) + h_1 \\ 0 \leq f(x)(t) &\leq D(x, y) \star f(x)(t + h_1) + h_1 \\ 0 \leq D(x, y) \star f(y)(t') &\leq f(y)(t' + h_2) + h_2 \\ 0 \leq f(y)(t') &\leq D(x, y) \star f(y)(t' + h_2) + h_2. \end{aligned}$$

From the second, the fourth inequalities and the fact that f is 1-Lipschitz, we get that for all $t \in]0, h_1^{-1}[$ and all $t' \in]0, h_2^{-1}[$

$$\begin{aligned} 0 \leq f(x)(t) \leq f(y)(t + h_1) + h_1 \\ 0 \leq f(y)(t') \leq f(x)(t' + h_2) + h_2. \end{aligned}$$

It follows that for all $s \in]0, \max(h_1, h_2)^{-1}[$ (a subset of $]0, \min(h_1^{-1}, h_2^{-1})[$)

$$\begin{aligned} 0 \leq f(x)(s) \leq f(y)(s + \max(h_1, h_2)) + \max(h_1, h_2) \\ 0 \leq f(y)(s) \leq f(x)(s + \max(h_1, h_2)) + \max(h_1, h_2). \end{aligned}$$

Thus, we have that $d_{\mathcal{L}}(f(x), f(y)) \leq \max(h_1, h_2)$ for all $h_1, h_2 > 0$ satisfying (4) and (5). This implies that

$$d_{\mathcal{L}}(f(x), f(y)) \leq \max(d_{\mathcal{L}}(D(x, y) \star f(x), f(x)), d_{\mathcal{L}}(D(x, y) \star f(y), f(y))),$$

and the conclusion. □

Let us now prove Theorem 1.

Proof of Theorem 1. The inequality at the left is a direct consequence of the definition of σ_D. To prove the inequality at the right, we use Lemma 4 noticing that

$$\mathcal{G}(G) := \{\delta_x / x \in G\} \subset Lip^1_\star(G, \Delta^+).$$

The second part of the theorem follows from Lemma 3 since, $y \in N_x(t)$ if and only if $d_{\mathcal{L}}(D(x,y), \mathcal{H}_0) < t$ for each $t > 0$ and $x, y \in G$. □

The notion of probabilistic distance naturally leads to associated metric concepts, such as Cauchy sequence, completeness, separability, density and compatness.

Definition 8. *A complete probabilistic metric space $(K, D, \star)$ is called compact if for all $t > 0$, the open cover $\{N_x(t) : x \in K\}$ has a finite subcover.*

Definition 9. *In a probabilistic metric space $(G, D, \star)$, a sequence $(z_n) \subset G$ is said to be a Cauchy sequence if for all $t \in \mathbb{R}$,*

$$\lim_{n,p \longrightarrow +\infty} D(z_n, z_p)(t) = \mathcal{H}_0(t).$$

(Equivalently, if $D(z_n, z_p) \xrightarrow{\text{w}} \mathcal{H}_0$ or $d_{\mathcal{L}}(D(z_n, z_p), \mathcal{H}_0) \to 0$, when $n, p \longrightarrow +\infty$). A probabilistic metric space $(G, D, \star)$ is said to be complete if every Cauchy sequence $(z_n) \subset G$ weakly converges to some $z_\infty \in G$, that is $\lim_{n\to+\infty} D(z_n, z_\infty)(t) = \mathcal{H}_0(t)$ for all $t \in \mathbb{R}$, we will briefly note $\lim_n D(z_n, z_\infty) = \mathcal{H}_0$.

Corollary 1. *Let $(G, D, \star)$ be a probabilistic metric space such that $\star$ is continuous. Then, the following assertions hold.*

(1) *$(G, D, \star)$ is a probabilistic complete metric space iff (G, σ_D) is a complete metric space.*

(2) *$(G, D, \star)$ is compact as probabilistic metric space iff (G, σ_D) is a compact metric space.*

(3) *$(G, D, \star)$ is separable as probabilistic metric space iff (G, σ_D) is separable metric space.*

Proof. It is a direct consequence of Theorem 1 using Lemma 3. □

Notice that several results in the litterature proved for probabilistic metric spaces could be easily deduced from Corollary 1 and Theorem 1. For instance, recall that a Baire space is a topological space such that every intersection of a countable collection of open dense sets is also dense. In [15], H. Sherwood proved that a complete Menger space under a continuous t-norm, equipped with the topology τ induced by the strong neighborhood system $\{N_x(t); x \in G, t > 0\}$ is a Baire space. Now, Theorem 1 expressing the fact that as soon as the triangle function is continuous then the induced topology is metrizable, we immediately obtain the following result.

Proposition 1. *Let $(G, D, \star)$ be a probabilistic complete metric space such that $\star$ is continuous. Let τ be the topology induced by strong neighborhood system $\{N_x(t); x \in G, t > 0\}$ (see Theorem 1). Then, (G, τ) is a Baire space.*

In the same spirit, we also easily recover the following proposition already proven by other means in [10, Theorem 2.2, Theorem 2.3].

Proposition 2. *Let $(K, D, \star)$ be a probabilistic metric space. Suppose that the triangle function $\star$ is continuous. Then,*

(1) $(K, D, \star)$ *is compact as probabilistic metric space iff every sequence of* K *has a convergent subsequence.*

(2) *If* $(K, D, \star)$ *is compact as probabilistic metric space, then it is separable.*

We end the section by showing that the metric σ_D is canonical in the following sens. We know that every (complete) metric space induce a probabilistic (complete) metric space. Indeed, if d is a (complete) metric on G and $\star$ is a triangle function on Δ^+ satisfying $\mathcal{H}_a \star \mathcal{H}_b = \mathcal{H}_{a+b}$ for all $a, b \in \mathbb{R}^+$ (see references [12] and [4]), then $(G, D, \star)$ is a probabilistic (complete) metric space, where

$$D(p,q) = \mathcal{H}_{d(p,q)}, \ \forall p, q \in G.$$

Using Proposition 3 below, we get that

$$\begin{aligned} d_{\mathcal{L}}(D(p,q), \mathcal{H}_0) \leq \sigma_D(p,q) := \sup_{z \in G} d_{\mathcal{L}}(D(p,z), D(z,q)) &= \sup_{z \in G} d_{\mathcal{L}}(\mathcal{H}_{d(p,z)}, \mathcal{H}_{d(z,q)}) \\ &\leq \sup_{z \in G} \min(1, |d(p,z) - d(z,q)|) \\ &= \min(1, d(p,q)) \\ &= d_{\mathcal{L}}(\mathcal{H}_{d(p,q)}, \mathcal{H}_0) \\ &= d_{\mathcal{L}}(D(p,q), \mathcal{H}_0) \end{aligned}$$

Thus, we have the equality

$$d_{\mathcal{L}}(D(p,q), \mathcal{H}_0) = \sigma_D(p,q) = \min(1, d(p,q)).$$

It follows that $\sigma_D(p,q) = d(p,q)$, for all $p, q \in G$ such that $d(p,q) \leq 1$. In particular, σ_D and d coincides if (G, d) is of diameter less than 1.

Proposition 3. *Let* $a, b \geq 0$. *Then,*

$$d_{\mathcal{L}}(\mathcal{H}_a, \mathcal{H}_b) = \min\left(1, |b-a|, \frac{1}{\min(a,b)}\right),$$

and, in particular,

$$d_{\mathcal{L}}(\mathcal{H}_a, \mathcal{H}_b) \leq \min(1, |b-a|) = d_{\mathcal{L}}(\mathcal{H}_{|b-a|}, \mathcal{H}_0). \tag{6}$$

Notice that the inequality (6) expresses the more general fact that, for all $\lambda > 0$ and for all F, $G \in \Delta^+$,

$$d_{\mathcal{L}}(\tau_\lambda F, \tau_\lambda G) \leq d_{\mathcal{L}}(F, G),$$

which is a consequence of the following property,

$$\forall \lambda > 0, \quad \left\{h > 0, [0, h^{-1}[\subset A^h_{F,G}\right\} \subset \left\{h > 0, [0, h^{-1}[\subset A^h_{\tau_\lambda F, \tau_\lambda G}\right\},$$

where $\tau_\lambda F(t) = F(t - \lambda)$. This contraction property is an equality for the standard Levy metric while Proposition (3) shows that it is not true for the modified version $d_{\mathcal{L}}$.

Proof. In this proof, we will assume without loss of generality that $a < b$ and use the shortened notation

$$A^h_{a,b} := A^h_{\mathcal{H}_a, \mathcal{H}_b} = \{t \geq 0, \mathcal{H}_a(t) \leq \mathcal{H}_b(t+h) + h\},$$

since in this case we have $\mathcal{H}_a \geq \mathcal{H}_b$. Notice that the inequality

$$H_a(t) \leq \mathcal{H}_b(t+h) + h$$

is immediate for $t \in [0,a]$, while if $t > a$ and since $h < 1$, it is equivalent to

$$\mathcal{H}_b(t+h) \geq 1 - h > 0,$$

that is $t + h > b$. As a consequence,

$$A^h_{a,b} = [0,a] \cup (]a,+\infty[\cap]b-h,+\infty[) = [0,a] \cup]\max(a, b-h), +\infty[.$$

We then have 2 cases :

- If $a > 1$, then for all $h \geq a^{-1}$, $[0,h^{-1}[\subset [0,a] \subset A^h_{a,b}$. In addition, if $h < a^{-1}$, then $[0,h^{-1}[\subset A^h_{a,b}$ if and only if $b - h \leq a$, that is $h \geq b - a$. This leads to

$$\left\{h > 0, [0,h^{-1}[\subset A^h_{a,b}\right\} = [a^{-1},+\infty[\cup [b-a,+\infty[= [\min(a^{-1}, b-a), +\infty[,$$

  hence in this case, $d_{\mathcal{L}}(\mathcal{H}_a, \mathcal{H}_b) = \min(a^{-1}, b-a) = \min(1, a^{-1}, b-a)$.
- If $a \leq 1$, we have $h^1 > a$ for all $h \in]0,1[$, hence $[0,h^{-1}[\subset A^h_{a,b}$ if and only if $b - h \leq a$, that is if $h \geq b - a$. It follows that

$$\left\{h > 0, [0,h^{-1}[\subset A^h_{a,b}\right\} = [1,+\infty] \cup (]0,1[\cap [\min(1,b-a),+\infty[) = [\min(1,b-a),+\infty[,$$

  hence, in this case, $d_{\mathcal{L}}(\mathcal{H}_a, \mathcal{H}_b) = \min(1, b-a) = \min(1, a^{-1}, b-a)$.

This concludes the proof. □

4. Fixed point and contraction

This section is divided on two subsections. In Subsection 4.1, we give two new fixed point theorems and in Subsection 4.2, we give some general examples of k-Lipschitz triangle functions constructed canonically from k-Lipschitz t-norms.

4.1. **Fixed point theorem.**

Let us start from the following probabilistic notion of contraction introduced by Hicks (see, [4]).

Definition 10. *Let $(G, D, \star)$ be a probabilistic metric space. A map $f : G \to G$ is said to be a C-contraction if there exists $q \in (0,1)$ such that for every $x, y \in G$ and every $t > 0$*

$$D(x,y)(t) > 1 - t \Longrightarrow D(f(x), f(y))(qt) > 1 - qt.$$

Lemma 5. *A map $f : G \to G$ is a C-contraction with constant q iff for all $x, y \in G$,*

$$d_{\mathcal{L}}(D(f(x), f(y)), \mathcal{H}_0) \leq q d_{\mathcal{L}}(D(x,y), \mathcal{H}_0).$$

Proof. From Lemma 3, we have that for every $x, y \in G$, $D(x,y)(t) > 1 - t$ if and only if $d_{\mathcal{L}}(D(x,y), \mathcal{H}_0) < t$. For every $\varepsilon > 0$, set $t_\varepsilon = d_{\mathcal{L}}(D(x,y), \mathcal{H}_0) + \varepsilon > 0$. Then, $D(x,y)(t_\varepsilon) > 1 - t_\varepsilon$. Suppose that f is a C-contraction, then we have that $D(f(x), f(y))(qt_\varepsilon) > 1 - qt_\varepsilon$ which is equivalent to

$$d_{\mathcal{L}}(D(f(x), f(y), \mathcal{H}_0) \leq qt_\varepsilon = q(d_{\mathcal{L}}(D(x,y), \mathcal{H}_0) + \varepsilon).$$

Sending ε to 0, we get $d_{\mathcal{L}}(D(f(x), f(y), \mathcal{H}_0) \leq q d_{\mathcal{L}}(D(x,y), \mathcal{H}_0)$. The converse is straightforward. □

Hicks proved that a C-contraction map in Menger space under the minimum t-norm $T_M(a,b) = \min(a,b)$ has a unique fixed point. We can find a extension of this result for generalised C-contraction in Menger space in [4]. We introduce the following new definition of contraction.

Definition 11. *Let $(G, D, \star)$ be a probabilistic metric space. Suppose that $\star$ is a continuous triangle function (hence uniformly continuous) and let $\omega_\star$ be a modulus of uniform continuity of $\star$. A map $f : G \to G$ is said to be a $\omega_\star$-contraction if there exists $q \in (0,1)$ such that for every $x, y \in G$*

$$\omega_\star[d_{\mathcal{L}}(D(f(x), f(y)), \mathcal{H}_0)] \leq q\omega_\star[d_{\mathcal{L}}(D(x,y), \mathcal{H}_0)].$$

Remark 2. Using Lemma 5, the notion of $\omega_\star$-contraction concides with the C-contraction, when the triangle function $\star$ is k-Lipschitz since in this case $\omega_\star(t) = kt$ for all $t \geq 0$ is a modulus of uniform continuity. Examples of k-Lipschitz triangle functions are given in Proposition 4. The original result of Hicks is a particular case corresponding to the 1-Lipschitz triangle function $\star_{T_M}$.

Using Theorem 1 and the Ekeland variational principle, we give below an extension of the result of Hicks in probabilistic metric spaces which are not necesarily Menger spaces, where the triangle function $\star$ is continuous. Notice that this result seems to be new even in the non probabilistic setting.

Theorem 2. *Let $(G, D, \star)$ be a probabilistic complete metric space, where $\star$ is continuous triangle function with modulus of uniform continuity $\omega_\star$. Let $f : G \to G$ be a $\omega_\star$-contraction with a constant of contraction $q \in (0,1)$. Then, f has a unique fixed point $x^* \in G$.*

Proof. By assumption, we have for all x, $y \in G$

$$\omega_\star(d_{\mathcal{L}}(D(f(x), f(y)), \mathcal{H}_0)) \leq q\omega_\star(d_{\mathcal{L}}(D(x,y), \mathcal{H}_0)).$$

Let us consider the function $\phi : (G, \sigma_D) \to \mathbb{R}$ defined by $\phi(x) = \omega_\star(d_{\mathcal{L}}(D(x, f(x)), \mathcal{H}_0))$ and prove that ϕ is continuous. Indeed, for $x, y \in G$, from the triangle inequality for $d_{\mathcal{L}}$, the definition of σ_D and Theorem 1 we have

$$\begin{aligned} |d_{\mathcal{L}}(D(x, f(x)), \mathcal{H}_0) - d_{\mathcal{L}}(D(y, f(y)), \mathcal{H}_0)| &\leq d_{\mathcal{L}}(D(x, f(x)), D(f(x), y)) \\ &\quad + d_{\mathcal{L}}(D(f(x), y), D(y, f(y))) \\ &\leq \sigma_D(x,y) + \sigma_D(f(x), f(y)) \\ &\leq \sigma_D(x,y) + \omega_\star(d_{\mathcal{L}}(D(f(x), f(y)), \mathcal{H}_0)) \\ &\leq \sigma_D(x,y) + q\omega_\star(d_{\mathcal{L}}(D(x,y), \mathcal{H}_0)). \end{aligned}$$

From the continuity of $\omega_\star$, we have that

$$d_{\mathcal{L}}(D(x,y), \mathcal{H}_0) \leq \sigma_D(x,y) \to 0 \Longrightarrow \omega_\star(d_{\mathcal{L}}(D(x,y), \mathcal{H}_0)) \to 0.$$

Thus, from the above inequalities, the function $x \mapsto d_{\mathcal{L}}(D(x, f(x)), \mathcal{H}_0)$ is continuous from (G, σ_D) into $\mathbb{R}$, and by composing it with the uniformly continuous function $\omega_\star$, we get that ϕ is continuous.

Now, by the Ekeland variational principle [3] (since (G, σ_D) is a complete metric space by Corollary 1), let $\varepsilon > 0$ and $u \in G$ such that $\phi(u) \leq \inf_G \phi + \varepsilon$. Then, for all $\lambda > 0$ there exists $v \in G$:

(i) $\phi(v) \leq \phi(u)$;

(ii) $\sigma_D(u, v) \leq \lambda$;

(iii) for all $x \in G$ $x \neq v$, $\phi(v) < \phi(x) + \frac{\varepsilon}{\lambda}\sigma_D(x, v)$.

Now, let us choose $\varepsilon < 1 - q$ and set $\lambda = 1$. Using Theorem 1 and (iii), we have that

$$\phi(v) \leq \phi(x) + \varepsilon\omega_\star(d_{\mathcal{L}}(D(x,v), \mathcal{H}_0)), \text{ for all } x \in G. \tag{7}$$

We claim that $x^* := f(v)$ is the unique fixed point of f. Indeed, we have

$$\omega_\star(d_{\mathcal{L}}(D(x^*, f(x^*)), \mathcal{H}_0)) = \omega_\star(d_{\mathcal{L}}(D(f(v), f(x^*)), \mathcal{H}_0)) \leq q\omega_\star(d_{\mathcal{L}}(D(v, x^*)). \quad (8)$$

By (7) with x^* and v, we have for all $x \in G$

$$\begin{aligned} \omega_\star(d_{\mathcal{L}}(D(v, x^*), \mathcal{H}_0)) &= \omega_\star(d_{\mathcal{L}}(D(v, f(v)), \mathcal{H}_0)) \\ &\leq \omega_\star(d_{\mathcal{L}}(D(x^*, f(x^*)), \mathcal{H}_0)) + \varepsilon\omega_\star(d_{\mathcal{L}}(D(x^*, v), \mathcal{H}_0)) \quad (9) \end{aligned}$$

Combining (8) and (9), we get

$$\omega_\star(d_{\mathcal{L}}(D(v, x^*), \mathcal{H}_0)) \leq (q + \varepsilon)\omega_\star(d_{\mathcal{L}}(D(v, x^*), \mathcal{H}_0)).$$

Since $q+\varepsilon < 1$, we obtain that $\omega_\star(d_{\mathcal{L}}(D(v, x^*), \mathcal{H}_0)) = 0$, which implies by Theorem 1 that $\sigma_D(v, x^*) = 0$, that is $x^* = v$. Thus, $f(x^*) = f(v) =: x^*$. The unicity of the fixed point x^* is immediate from $q < 1$. □

Applying the Banach fixed point we give the following extenstion of Hicks's result with an estimation of convergence of sequences $x_{n+1} = f(x_n)$. Note that we recover the Hicks's result with $\star = \star_{T_M}$ which is k-Lipschitz, with $k = 1$.

Theorem 3. *Let $(G, D, \star)$ be a probabilistic complete metric space, where $\star$ is k-Lipschitz triangle function ($k \geq 1$). Let $f : G \to G$ be a C-contraction with a constant of contraction $q \in]0, \frac{1}{k}[$. Then, f has a unique fixed point $x^* \in G$. Moreover, every sequence (x_n) of G such that $x_{n+1} = f(x_n)$, satisfies: for every $t > 0$*

$$D(x_1, x_0)(t) > 1 - t \Longrightarrow D(x_n, x^*)(\frac{k(kq)^n}{1 - kq}t) > 1 - \frac{k(kq)^n}{1 - kq}t,$$

or equivalently,

$$d_{\mathcal{L}}(D(x_n, x^*), \mathcal{H}_0) \leq \frac{k(kq)^n}{1 - kq} d_{\mathcal{L}}(D(x_1, x_0), \mathcal{H}_0).$$

In particular, $d_{\mathcal{L}}(D(x_n, x^*), \mathcal{H}_0) \to 0$, when $n \to +\infty$.

Proof. By Lemma 5, we have that $d_{\mathcal{L}}(D(f(x), f(y), \mathcal{H}_0) \leq q d_{\mathcal{L}}(D(x, y), \mathcal{H}_0)$, for all $x, y \in G$. Using Theorem 1, we get

$$\sigma_D(f(x), f(y)) \leq qk\sigma_D(x, y).$$

Since $qk < 1$, we can apply the Banach fixed point theorem in the complete metric space (G, σ_D). Thus, we obtain a unique fixed point x^* such that, for all $n \in \mathbb{N}$

$$\sigma_D(x_n, x^*) \leq \frac{(kq)^n}{1 - kq}\sigma_D(x_1, x_0).$$

Using again Theorem 1 (the k-Lipschitz part) we give

$$d_{\mathcal{L}}(D(x_n, x^*), \mathcal{H}_0) \leq \frac{k(kq)^n}{1 - kq} d_{\mathcal{L}}(D(x_1, x_0), \mathcal{H}_0),$$

which is equivalent by Lemma 5 to: for all $t > 0$

$$D(x_1, x_0)(t) > 1 - t \Longrightarrow D(x_n, x^*)(\frac{k(kq)^n}{1 - kq}t) > 1 - \frac{k(kq)^n}{1 - kq}t.$$

□

4.2. **k-Lipschitz triangle function.** One of the standard way to construct triangle function goes in the following way. We refer to [4] for more details.

Definition 12. *We denote by $\mathcal{L}$ the set of all binary operators L on $[0,+\infty[$ which satisfy the following conditions:*
(i) *L maps $[0,+\infty[^2$ to $[0,+\infty[$*
(ii) *L is non-deceasing in both coordinate*
(iii) *L is continuous on $[0,+\infty[^2$.*

For a t-norm T, we define the operation $\star_{T,L}$ from $\Delta^+ \times \Delta^+$ to Δ^+ as follows: for every $F, G \in \Delta^+$ and every $t \geq 0$

$$(F \star_{T,L} G)(t) = \sup_{L(u,v)=t} T(F(u), G(v)).$$

In the speciale case where $L(u,v) = u + v$ we obtain $\star_{T,L} = \star_T$.

Theorem 4. ([4, Theorem 2.15]) *if T is a left-continuous t-norm and $L \in \mathcal{L}$ is commutative, associative, has 0 as identity and satisfy the condition*

$$\text{if } u_1 < u_2 \text{ and } v_1 < v_2 \text{ then } L(u_1, v_1) < L(u_2, v_2),$$

then, $\star_{T,L}$ is a triangle function.

The above theorem works for example with $L(u,v) := L_+(u,v) = u + v$ or $L(u,v) := L_M(u,v) = \max(u,v)$.

Another way to construct a triangle function from a t-norm T is the use the t-conorm $T^*(u,v) = 1 - T(1-u, 1-v)$ as follows : for every $F, G \in \Delta^+$ and for every $s > 0$

$$(F \star_{T^*} G)(s) = \inf_{u+v=s} T^*(F(u), G(v)).$$

Recall that a t-norm $T : [0,1] \times [0,1] \to [0,1]$ is k-Lipschitz if there exists $k \in [0,+\infty[$ such that, for all $a, b, c, d \in [0,1]$, we have

$$|T(a,b) - T(c,d)| \leq k(|a-c| + |b-d|).$$

Since $T(x,1) = x$, we necessarily have that $k \geq 1$. Note also that the minimum t-norm $T_M(a,b) := \min(a,b)$ is 1-Lipschitz. Other examples of k-Lipschitz t-norms are studied in [7, 8, 9]. In order to give examples of k-Lipschitz triangle functions in Proposition 4, we need the following lemma.

Lemma 6. *Let $T : [0,1] \times [0,1] \to [0,1]$ be a k-Lipschitz t-norm. Then, for every $(a,b), (c,d) \in [0,1] \times [0,1]$ and every $h \in [0,+\infty[$ such that $a \leq c + h_1$ and $b \leq d + h_2$ we have that*

$$T(a,b) - T(c,d) \leq k(h_1 + h_2)$$
$$T^*(a,b) - T^*(c,d) \leq k(h_1 + h_2).$$

Proof. Four cases are discussed.

case 1. If $a \leq c$ and $b \leq d$. In this case, since T is a t-norm, then

$$T(a,b) - T(c,d) \leq 0 \leq k(h_1 + h_2).$$

case 2. If $a \leq c$ and $b \geq d$. In this case, since T is a t-norm, then $T(a,b) \leq T(c,b)$ and so since it is k-Lipschitz we have that

$$T(a,b) - T(c,d) \leq T(c,b) - T(c,d) \leq k|b-d| = k(b-d) \leq kh_1 \leq k(h_1 + h_2).$$

case 3. If $a \geq c$ and $b \leq d$. This case is similar to case 2.
case 4. If $a \geq c$ and $b \geq d$. In this case, since T is k-Lipschitz we have that

$$T(a,b) - T(c,d) \leq k(|a-c| + |b-d|) = k(a-c+b-d) \leq k(h_1+h_2).$$

The case of T^* comes easily from the case of T. □

In the following proposition, we consider the cases where $L(u,v) := L_+(u,v) = u+v$ and $L(u,v) := L_M(u,v) = \max(u,v)$.

Proposition 4. *Let $T : [0,1]\times[0,1] \to [0,1]$ be a k-Lipschitz t-norm ($k \geq 1$). Then, the triangle functions $\star_T$; $\star_{T,L_M}$; $\star_{T^*}$ and $\mathbb{T}$ are k-Lipschitz , where for all $F, G \in \Delta^+$ and all $s > 0$*

$$(F \star_T G)(s) = \sup_{u+v=s} T(F(u), G(v)),$$

$$(F \star_{T^*} G)(s) = \inf_{u+v=s} T^*(F(u), G(v)),$$

$$(F \star_{T,L_M} G)(s) = \sup_{\max(u,v)=s} T(F(u), G(v)),$$

$$\mathbb{T}(F,G)(s) = T(F(s), G(s)).$$

Proof. We give the prove for $\star_T$, the technique is similar for the other triangle functions. Let $F, F', G, G' \in \Delta^+$. Let $h_1, h_2 > 0$ be such that

$$[0, h_1^{-1}[\subset A^{h_1}_{F,F'} \cap A^{h_1}_{F',F} \text{ and } [0, h_2^{-1}[\subset A^{h_2}_{G,G'} \cap A^{h_2}_{G',G}, \tag{10}$$

meanning that for all $t \in]0, h_1^{-1}[$ and all $t' \in]0, h_2^{-1}[$ we have:

$$\begin{aligned} 0 \leq F(t) &\leq F'(t+h_1) + h_1 \\ 0 \leq F'(t) &\leq F(t+h_1) + h_1 \\ 0 \leq G(t') &\leq G'(t'+h_2) + h_2 \\ 0 \leq G'(t') &\leq G(t'+h_2) + h_2. \end{aligned}$$

Thus, combining the first and the third (resp. the second and the fourth) inequalities, and using Lemma 6, we have that for every $u, v \in]0, \max(h_1,h_2)^{-1}[(\subset]0, \min(h_1^{-1}, h_2^{-1})[)$

$$\begin{aligned} 0 \leq T(F(u), G(v)) &\leq T(F'(u+h_1), G'(v+h_2)) + k(h_1+h_2) \\ 0 \leq T(F'(u), G'(v)) &\leq T(F(u+h_1), G(v+h_2)) + k(h_1+h_2) \end{aligned}$$

Let $s \in]0, \max(h_1,h_2)^{-1}[$, taking the supremum over $0 \leq u, v$ such that $u+v=s$ in the above inequalities with the fact that $k \geq 1$, we get

$$\begin{aligned} 0 \leq (F \star_T G)(s) &\leq (F' \star_T G')(s + (h_1+h_2)) + k(h_1+h_2) \\ &\leq (F' \star_T G')(s + k(h_1+h_2)) + k(h_1+h_2) \\ 0 \leq (F' \star_T G')(s) &\leq (F \star_T G)(s + (h_1+h_2)) + k(h_1+h_2) \\ &\leq (F \star_T G)(s + k(h_1+h_2)) + k(h_1+h_2) \end{aligned}$$

This shows that for all $h_1, h_2 > 0$ satisfying (10), we have

$$d_{\mathcal{L}}(F \star_T G, F' \star_T G') \leq k(h_1+h_2).$$

Thus, taking the infinimum over h_1 and h_2, we get

$$d_{\mathcal{L}}(F \star_T G, F' \star_T G') \leq k(d_{\mathcal{L}}(F,F') + d_{\mathcal{L}}(G,G')).$$

□

5. Probabilistic Arzela-Ascoli type theorem

This section is divided on two subsections. In subsection 5.1 we give some general definitions of probabilistic function spaces and in subsection 5.2, we give the main result of this section, a probabilistic Arzela-Ascoli type theorem.

5.1. **The space of continuous and functions.**

We are going to define continuity of functions defined from a probabilistic metric space $(G, D, \star)$ to a (deterministic) metric space (F, d_F).

Definition 13. *Let $(G, D, \star)$ be a probabilistic metric space, (F, d_F) be a metric space and let f be a function $f : (G, D, \star) \longrightarrow (F, d_F)$. We say that f is (probabilistic) continuous at $z \in G$ if $d_F(f(z_n) f(z)) \to 0$ whenever $D(z_n, z) \xrightarrow{\mathrm{w}} \mathcal{H}_0$ (equivalently $d_{\mathcal{L}}(D(z_n, z), \mathcal{H}_0) \to 0$). We say that f is continuous if f is continuous at each point $z \in G$.*

By $C_\star(G, F)$ we denote the space of all (probabilistic) continuous functions $f : (G, D, \star) \longrightarrow (F, d_F)$. By $C(G, F)$ we denote the space of all (deterministic) continuous functions $f : (G, \sigma_D) \longrightarrow (F, d_F)$. We both equip the spaces $C_\star(G, F)$ and $C(G, F)$ with the uniform metric

$$d_\infty(f, g) := \sup_{x \in G} d_F(f(x), g(x))$$

As in the standard case, the completeness of $C_\star(G, F)$ only relies on the completeness of the arrival space.

Proposition 5. *Let $(G, D, \star)$ be a probabilistic metric space (here $\star$ is not assumed to be continuous) and (F, d_F) be a complete metric space. Then, the space $(C_\star(G, F), d_\infty)$ is a complete metric space.*

Proof. Let (f_n) be a Cauchy sequence in $(C_\star(G, F), d_\infty)$. In particular, for each $x \in G$, $(f_n(x))$ is Cauchy in (F, d_F) which is complete. Thus, there exists a function $f : G \longrightarrow F$ such that the sequence (f_n) pointwise converges to f on G. It is easy to see that in fact (f_n) uniformly converges to f, since it is Cauchy sequence in $(C_\star(G, F), d_\infty)$. We need to prove that f is a continuous function from $(G, D, \star)$ into (F, d_F). Let $x \in G$ and (x_k) be a sequence such that $d_{\mathcal{L}}(D(x_k, x), \mathcal{H}_0) \longrightarrow 0$, when $k \longrightarrow +\infty$. For all $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$n \geq N_\varepsilon \Longrightarrow d_\infty(f_n, f) := \sup_{x \in G} d_F(f_n(x), f(x)) \leq \varepsilon \tag{11}$$

Using the continuity of f_{N_ε}, we have that there exists $\eta(\varepsilon) > 0$ such that

$$d_{\mathcal{L}}(D(x_k, x), \mathcal{H}_0) \leq \eta(\varepsilon) \Longrightarrow d_F(f_{N_\varepsilon}(x_k), f_{N_\varepsilon}(x)) \leq \varepsilon \tag{12}$$

Using (11) and (12), we have that

$$\begin{aligned} d_F(f(x_k), f(x)) &\leq d_F(f(x_k), f_{N_\varepsilon}(x_k)) + d_F(f_{N_\varepsilon}(x_k), f_{N_\varepsilon}(x)) + d_F(f_{N_\varepsilon}(x), f(x)) \\ &\leq 3\varepsilon \end{aligned}$$

This shows that f is continuous on G. Finally, we proved that every Cauchy sequence (f_n) uniformly converges to a continuous function f. In other words, the space $(C_\star(G, F), d_\infty)$ is complete. □

Since, for all $x,\, y \in G$,

$$d_{\mathcal{L}}(D(x,y), \mathcal{H}_0) \le \sigma_D(x,y),$$

We have in general that $C_\star(G,F) \subset C(G,F)$. Assuming the continuity of the triangle function $\star$, we obtain the equality.

Proposition 6. *Let $(G,D,\star)$ be a probabilistic metric space and (F,d) be a metric space. Suppose that $\star$ is continuous. Then, we have $C_\star(G,F) = C(G,F)$. In the case where $(F,d_F) = (\Delta^+, d_{\mathcal{L}})$, we also have that $Lip^1_\star(G,\Delta^+) \subset C_\star(G,\Delta^+)$.*

Proof. Thanks to Theorem 1, we have, for all $x,\, y \in G$,

$$\sigma_D(x,y) \le \omega_\star\left(d_{\mathcal{L}}(D(x,y)), \mathcal{H}_0\right),$$

providing the equality between $C_\star(G,F)$ and $C(G,F)$. For the second part of the statement, we use Lemma 4 to ensure that, for any $f \in Lip^1_\star(G,\Delta^+)$ and for any $x,\, y \in G$,

$$d_{\mathcal{L}}(f(x), f(y)) \le \sup_{g \in Lip^1_\star(G,\Delta^+)} [d_{\mathcal{L}}(g(x), g(y))] \le \omega_\star\left(d_{\mathcal{L}}(D(x,y)), \mathcal{H}_0\right),$$

which gives the conclusion. □

Remark 3. We do not know if $Lip^1_\star(G,\Delta^+) \subset C_\star(G,\Delta^+)$ when $\star$ is not continuous.

5.2. **Arzela-Ascoli type theorem for the space $Lip^1_\star(K,\Delta^+)$.** The following proposition gives a canonical way to build probabilistic 1-Lipschitz maps from $(G,D,\star)$ into Δ^+.

Definition 14. *A triangle function $\star$ is said to be sup-continuous (see for instance* [2]*) if for all nonempty set I and all familly $(F_i)_{i\in I}$ of distributions in Δ^+ and all $L \in \Delta^+$, we have*

$$\sup_{i\in I}(F_i \star L) = \sup_{i\in I}(F_i) \star L.$$

Proposition 7. *Let $(G,D,\star)$ be a probabilistic metric space such that $\star$ is sup-continuous. Let $f : (G,D,\star) \longrightarrow \Delta^+$ be any map and A be any no-empty subset of G. Then, the map $\tilde{f}_A(x) := \sup_{y\in A}[f(y) \star D(x,y)]$, for all $x \in G$ is a probabilistic 1-Lipschitz map and we have $\tilde{f}_A(x) \ge f(x)$, for all $x \in A$.*

Proof. The proof is similar to the standard inf-convolution construction. The fact that $\tilde{f}_A(x) \ge f(x)$ for all $x \in A$ is immediate from the definition of $\tilde{f}_A$. Let us now prove that it is probabilistic 1-Lipschiptz. Let $x,\, y \in G$. Then, for all $z \in A$, we have

$$\begin{aligned}\tilde{f}_A(y) = \sup_{z\in A}[f(z) \star D(y,z)] &\ge f(z) \star D(y,z)\\ &\ge f(z) \star (D(y,x) \star D(x,z)) = (f(z) \star D(x,z)) \star D(y,x).\end{aligned}$$

We get the conclusion by taking the supremum with respect to $z \in A$ and using the sup-continuity of $\star$. □

Let us now recall the following result from [1].

Proposition 8. *(*[1, Proposition 3.5]*) Let $(F_n), (L_n), (K_n) \subset (\Delta^+, \star)$. Suppose that*

(a) *the triangle function $\star$ is continuous,*
(b) *$F_n \xrightarrow{w} F$, $L_n \xrightarrow{w} L$ and $K_n \xrightarrow{w} K$.*

(c) *for all* $n \in \mathbb{N}$, $F_n \star L_n \leq K_n$.

Then, $F \star L \leq K$.

Lemma 7. *Let* $(K, D, \star)$ *be a probabilistic compact metric space and* (f_n) *be a sequence of probabilistic* 1*-Lipschitz maps. Suppose that there exists a function* f *defined from* K *into* Δ^+ *such that, for all* $x \in K$, $d_{\mathcal{L}}(f_n(x), f(x)) \longrightarrow 0$, *as* $n \to +\infty$. *Then,* f *is (probabilistic)* 1*-Lipschitz on* K *and* (f_n) *converges uniformly to* f, *that is,* $d_\infty(f_n, f) \longrightarrow 0$, *as* $n \to +\infty$.

Proof. Since each f_n is a 1-Lipschitz map, we have for all $x, y \in L$ and for all $n \in \mathbb{N}$:

$$D(x, y) \star f_n(x) \leq f_n(y)$$

Using Proposition 8, we get that for all $x, y \in L$

$$D(x, y) \star f(x) \leq f(y)$$

In other words, f is 1-Lipschitz maps on L (Note that up to now, we have not needed to use the compactness of K).

Now, let $\varepsilon > 0$ and, using Lemma 4, let $\eta(\varepsilon)$ be the uniform modulus of equicontinuity for the set $Lip^1_\star(K, \Delta^+)$. Since $(K, D, \star)$ is compact, there exists a finite set A such that $K = \cup_{a \in A} N_a(\eta(\varepsilon))$. Since $d_{\mathcal{L}}(f_n(a), f(a)) \longrightarrow 0$, as $n \to +\infty$ for all $a \in A$. Then, for each $a \in A$, there exists $P_a \in \mathbb{N}$ such that

$$n \geq P_a \Longrightarrow d_{\mathcal{L}}(f_n(a), f(a)) \quad \leq \quad \varepsilon$$

Since A is finite, we have that

$$n \geq \max_{a \in A} P_a \Longrightarrow \sup_{a \in A} d_{\mathcal{L}}(f_n(a), f(a)) \quad \leq \quad \varepsilon$$

Thus, for all $x \in K = \cup_{a \in A} N_a(\eta(\varepsilon))$, there exists $a \in A$ such that $x \in N_a(\eta(\varepsilon))$ and so we have that for all $n \geq \max_{a \in A} P_a$:

$$\begin{aligned} d_{\mathcal{L}}(f_n(x), f(x)) &\leq d_{\mathcal{L}}(f_n(x), f_n(a)) + d_{\mathcal{L}}(f_n(a), f(a)) + d_{\mathcal{L}}(f(a), f(x)) \\ &\leq 3\varepsilon. \end{aligned}$$

In other words,

$$n \geq \max_{a \in A} P_a \Longrightarrow d_\infty(f_n, f) := \sup_{x \in K} d_{\mathcal{L}}(f_n(x), f(x)) \quad \leq \quad 3\varepsilon$$

□

We give now our main result of this section. For the classical Arzela-Ascoli theorem we refer to the book of L. Schwartz, Analyse I, *"Thorie des ensembles et Topologie"*, page 346.

Theorem 5. *Let* $(K, D, \star)$ *be a probabilistic complete metric space such that* $\star$ *is continuous. Then, the following assertions are equivalent.*

(1) $(K, D, \star)$ *is compact.*
(2) *The metric space* $(Lip^1_\star(K, \Delta^+), d_\infty)$ *is compact (or equivalently,* $Lip^1_\star(K, \Delta^+)$ *is a compact subset of* $(C_\star(K, \Delta^+), d_\infty) = (C(K, \Delta^+), d_\infty)$*).*

Proof. • (1) $\Longrightarrow$ (2) Suppose that $(K, D, \star)$ is compact, equivalently (K, σ_D) is compact by Corollary 1. Using Lemma 4 and Theorem 1, the set $Lip^1_\star(G, \Delta^+)$ is uniformly equicontinuous with respect to the metric σ_D. Moreover, $(\Delta^+, d_{\mathcal{L}})$ is compact, hence $Lip^1_\star(G,\Delta^+)$ is relatively compact in $(C(K,\Delta^+),d_\infty)$ by Arzela-Ascoli theorem. On the other hand, by Lemma 7, the set $Lip^1_\star(G,\Delta^+)$ is closed in $(C(G,\Delta^+),d_\infty)$. Hence it is compact.

• (2) $\Longrightarrow$ (1) Suppose that $(Lip^1_\star(K,\Delta^+),d_\infty)$ is compact. Let (x_n) be a sequence of K. We need to prove that (x_n) has a convergent subsequence. Consider the sequence (δ_{x_n}) of 1-Lipschitz maps, defined by $\delta_{x_n}: x\mapsto D(x_n,x)$ for each $n\in\mathbb{N}$. By assumption, there exists a subsequence $(\delta_{x_{\varphi(n)}})$ that converges uniformly to some 1-Lipschitz map, in particular it is a Cauchy sequence. In other words, we have

$$\lim_{p,q\longrightarrow+\infty}\sup_{x\in K} d_{\mathcal{L}}(\delta_{x_{\varphi(p)}}(x),\delta_{x_{\varphi(q)}}(x))=0.$$

In particular we have

$$\lim_{p,q\longrightarrow+\infty} d_{\mathcal{L}}(\delta_{x_{\varphi(p)}}(x_{\varphi(q)}),\mathcal{H}_0)=0,$$

or equivalently,

$$\lim_{p,q\longrightarrow+\infty} d_{\mathcal{L}}(D(x_{\varphi(p)},x_{\varphi(q)}),\mathcal{H}_0)=0.$$

This shows that the sequence $(x_{\varphi(n)})$ is Cauchy in (K,σ_D) (see Theorem 1). Thus, the sequence $(x_{\varphi(n)})$ converges to some point $x\in K$ for the metric σ_D, since (K,σ_D) is complete. Hence, (K,σ_D) is compact, equivalently, $(K,D,\star)$ is compact. This ends the proof. □

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