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Mohammed Bachir. THE SPACE OF PROBABILISTIC 1-LIPSCHITZ MAPS. 2017. hal-01673497v1

**HAL Id: hal-01673497**

**<https://paris1.hal.science/hal-01673497v1>**

Preprint submitted on 30 Dec 2017 (v1), last revised 16 Jan 2018 (v2)

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# THE SPACE OF PROBABILISTIC 1-LIPSCHITZ MAPS

MOHAMMED BACHIR

**ABSTRACT.** We introduce and study a natural notion of probabilistic 1-Lipschitz maps. We prove that the space of all probabilistic 1-Lipschitz maps defined on a probabilistic metric space  $G$  is also a probabilistic metric space. Moreover, when  $G$  is a group, then the space of all probabilistic 1-Lipschitz maps defined on  $G$  can be endowed with a monoid structure. Then, we characterize the probabilistic invariant complete Menger groups by the space of all probabilistic 1-Lipschitz maps in the spirit of the classical Banach-Stone theorem.

## 1. INTRODUCTION

The general concept of probabilistic metric spaces was introduced by K. Menger, who dealt with probabilistic geometry [11], [12], [13]. The decisive influence on the development of the theory of probabilistic metric spaces is due to B. Schweizer and A. Sklar and their coworkers in several papers [16], [17], [18], [19], see also [21]. For more informations about this theory we refer to the excellent monograph [15].

The purpose of this paper is to make a new and complementary contributions to this theory. We introduce, in a natural way, the concept of probabilistic 1-Lipschitz maps defined from a probabilistic metric space  $G$  into the set of all distribution functions that vanish at 0, classically denoted by  $\Delta^+$ . This notion, allows in particular to obtain a probabilistic analogue of the classical result of E. Mac Shane [10] on the extension of Lipschitz maps from a subset to the whole space (Theorem 2). We study the properties of the space of all probabilistic 1-Lipschitz maps denoted by  $Lip_*^1(G, \Delta^+)$  in the paper. As in the classical determinist analysis, where Lipschitz maps play important role, we show that it is the same for the space of probabilistic 1-Lipschitz maps. We will start with relatively basic property by showing that this space has also a probabilistic metric structure in which the probabilistic metric space  $G$  embeds isometrically as probabilistic metric space. In the case where  $G$  is endowed with a group structure, we prove that we can also equip the space  $Lip_*^1(G, \Delta^+)$  with a natural "**Sup-Convolution**" operation denoted by  $\odot$  so that  $(Lip_*^1(G, \Delta^+), \odot)$  become a probabilistic metric monoid. Let us note in

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*Date:* 27/12/217.

*1991 Mathematics Subject Classification.* 54E70, 47S50, 46S50.

passing that the monoid  $(\Delta^+, \tau)$  (equipped with a triangle function  $\tau$  and having  $\mathcal{H}_0$  as identity element) himself coincides with  $(Lip_\star^1(\{\mathcal{H}_0\}, \Delta^+), \odot)$  and  $\odot$  coincides with  $\tau$  in this case. This shows, in particular, the naturalness of the space  $(Lip_\star^1(G), \Delta^+), \odot)$  introduced in this paper. Then, we prove in a general setting our main results, which implies that the probabilistic invariant complete metric group structure of  $G$  is completely determined by the probabilistic metric monoid structure of  $(Lip_\star^1(G, \Delta^+), \odot)$  (see the main results Theorem 4, Theorem 5 and Corollary 2). Our results apply in particular to probabilistic invariant complete Menger groups (Corollary 2). This gives a probabilistic version of the well known Banach-Stone theorem and extend some results obtained recently by the author for classical invariant metric groups in [2], [3] and [4]. Recall that the classical Banach-Stone theorem asserts that a compact Hausdorff space  $K$  is completely determined by the Banach structure of the space  $(C(K), \|\cdot\|_\infty)$  of all real-valued continuous functions equipped with the sup-norm. In other words, two Hausdorff compact spaces  $K$  and  $L$  are homeomorphic if and only if  $(C(K), \|\cdot\|_\infty)$  and  $(C(L), \|\cdot\|_\infty)$  are isometrically isomorphic as Banach spaces. Finally, note that several authors was interested on the extention of classical results obtained in metric spaces to the general framework of probabilistic metric spaces. For example, the theory of fixed points has several extensions in the setting of probabilistic metric spaces (see [8] and [9]). Also, the classical result of Mazur-Ulam on isometries in normed vector spaces has been extended to the probabilistic normed spaces (see for instance [5], [6]).

This paper is organized as follows. In Section 2, we recall some classical and known result about probabilistic metric spaces. In Section 3 and Section 4 we deal with some basic properties as follows: In Section 3, we introduce the notion of probabilistic 1-Lipschitz maps defined on a probabilistic metric space and give an analogue to the MC Shane's result on the extension of Lipschitz maps to the whole space. We prove that the space of all 1-Lipschitz maps is also a probabilistic metric space. In Section 3, we work in the framework of probabilistic invariant metric group. We introduce the concept of probabilistic Sup-Convolution and prove that the space of all 1-Lipschitz maps endowed with the Sup-Convolution operation has a monoid structure, in which the initial probabilistic invariant metric group embeeds isometrically as group. In Section 5, we give our main results Theorem 4, Theorem 5 and Corollary 2.

## 2. CLASSICAL NOTIONS OF PROBABILISTIC METRIC SPACES.

In this section, we recall somme general well know definitions and concepts about probabilistic metric spaces. All these concepts can be found in [8] and [15].

A distribution function is a function  $F : [-\infty, +\infty] \longrightarrow [0, 1]$  which is nondecreasing and left-continuous with  $F(-\infty) = 0$ ;  $F(+\infty) = 1$ . The set

of all distribution functions such that  $F(0) = 0$  will be denoted by  $\Delta^+$ . For  $F, G \in \Delta^+$ , the relation  $F \leq G$  is meant by  $F(t) \leq G(t)$ , for all  $t \in \mathbb{R}$ . For all  $a \in \mathbb{R}$ , the distribution  $\mathcal{H}_a$  is defined as follow

$$\mathcal{H}_a(t) = \begin{cases} 0 & \text{for all } t \leq a \\ 1 & \text{for all } t > a \end{cases}$$

For  $a = +\infty$ ,

$$\mathcal{H}_\infty(t) = \begin{cases} 0 & \text{for all } t \in [-\infty, +\infty[ \\ 1 & \text{for all } t = +\infty \end{cases}$$

It is well known that  $(\Delta^+, \leq)$  is a complete lattice with a minimal element  $\mathcal{H}_\infty$  and the maximal element  $\mathcal{H}_0$ . Thus, for any nonempty set  $I$  and any family  $(F_i)_{i \in I}$  of distributions in  $\Delta^+$ , the function  $F = \sup_{i \in I} F_i$  is also an element of  $\Delta^+$ .

**Definition 1.** (see [15], [8]) *A triangle function  $\tau$  is a binary operation on  $\Delta^+$  that is commutative, associative, and non-decreasing in each place, and has  $\mathcal{H}_0$  as identity. In other words,  $\tau$  satisfies the axioms (i)-(v):*

- (i)  $\tau(F, L) \in \Delta^+$  for all  $F, L \in \Delta^+$ .
- (ii)  $\tau(F, L) = \tau(L, F)$  for all  $F, L \in \Delta^+$ .
- (iii)  $\tau(F, \tau(L, K)) = \tau(\tau(F, L), K)$ , for all  $F, L, K \in \Delta^+$ .
- (iv)  $\tau(F, \mathcal{H}_0) = F$  for all  $F \in \Delta^+$ .
- (v)  $F \leq L \implies \tau(F, K) \leq \tau(L, K)$  for all  $F, L, K \in \Delta^+$ .

In this paper, we assume also that  $\tau$  is sup-continuous (see [7]) that is is a triangle function satisfying the following additional axiom (see Example 1 below):

- (vi) If  $I$  is a nonempty set,  $(F_i)_{i \in I}$  a family of distributions in  $\Delta^+$  and  $G \in \Delta^+$ , then  $\sup_{i \in I} \tau(F_i, L) = \tau(\sup_{i \in I} F_i, L)$ .

For simplicity of notations, in what follows, the triangle function  $\tau$  will be denoted by the binary operation  $\star$  as follows:

$$\tau(L, K) := L \star K.$$

It follows from the axioms (i)-(vi) that  $(\Delta^+, \star)$  is in particular an abelian monoid having  $\mathcal{H}_0$  as identity element.

**Definition 2.** *We say that the law  $\star$  is continuous at  $(F, L) \in \Delta^+ \times \Delta^+$  if we have  $\lim_{n \rightarrow +\infty} F_n \star L_n(t) = F \star L(t)$  for all  $t \in \mathbb{R}$  point of continuity of  $F \star L$ , whenever  $\lim_{n \rightarrow +\infty} F_n(t) = F(t)$  for all  $t \in \mathbb{R}$  point of continuity of  $F$  and  $\lim_{n \rightarrow +\infty} L_n(t) = L(t)$  for all  $t \in \mathbb{R}$  point of continuity of  $L$ .*

**Definition 3.** (see [15], [8]) *A triangular norm (t-norm for short) is a binary operation on the unit interval  $[0, 1]$ , i.e., a function  $T : [0, 1] \times [0, 1] \longrightarrow [0, 1]$  such that for all  $x, y, z \in [0, 1]$  the following four axioms are satisfied:*

- (T1)  $T(x, y) = T(y, x)$  (commutativity);

- (T2)  $T(x, T(y, z)) = T(T(x, y), z)$  (associativity);  
 (T3)  $T(x, y) \leq T(x, z)$  whenever  $y \leq z$  (monotonicity);  
 (T4)  $T(x, 1) = x$  (boundary condition).

Several basic examples of left-continuous  $t$ -norm are given in [8] and [15]. We give below a classical continuous triangle functions.

*Examples 1.* (see [8] and [15]) Let  $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$  be a left-continuous  $t$ -norm, then the operation  $\star_T$  defined for all  $F, L \in \Delta^+$  and for all  $t \in \mathbb{R}$  by

$$(1) \quad (F \star_T L)(t) := \sup_{s+u=t} T(F(s), L(u))$$

is continuous at each point  $(F, L) \in \Delta^+ \times \Delta^+$  and satisfies the axioms (i)-(vi).

*Remark 1.* In all this paper, we assume that  $\star$  is a continuous triangle function satisfying the axioms (i)-(vi).

**Definition 4.** (see Example 2 below) Let  $G$  be a set and let  $D : G \times G \rightarrow (\Delta^+, \star, \leq)$  be a map. We say that  $(G, D, \star)$  is a probabilistic metric space if the following axioms (i)-(iii) hold:

- (i)  $D(p, q) = \mathcal{H}_0$  iff  $p = q$ .
- (ii)  $D(p, q) = D(q, p)$  for all  $p, q \in G$
- (iii)  $D(p, q) \star D(q, r) \leq D(p, r)$  for all  $p, q, r \in G$

If moreover the set  $(G, \cdot)$  is a monoid, then we say that  $(G, \cdot, D, \star)$  is a probabilistic metric monoid and if  $(G, \cdot)$  is group and satisfies the following additional axiom,

- (iv)  $D(pr, qr) = D(rp, rq) = D(p, q)$  for all  $p, q, r \in G$
- then, we say that  $(G, \cdot, D, \star)$  is an invariant probabilistic metric group.

**Definition 5.** Two probabilistic metric spaces  $(G, D, \star)$  and  $(G', D', \star)$  (resp. monoids, groups) are said to be isometric (resp. isometrically isomorphic), if there exists a bijective map (resp. isomorphism of monoids, isomorphism of groups)  $\mathcal{I} : G \rightarrow G'$  such that  $D'(\mathcal{I}(x), \mathcal{I}(y)) = D(x, y)$  for all  $x, y \in G$ .

**Definition 6.** Let  $(G, D, \star)$  be a probabilistic metric space and  $\star = \star_T$ , where

$$(K \star_T L)(t) = \sup_{u+v=t} T(K(u), L(v))$$

for a  $t$ -norm  $T$ . Then  $(G, D, \star)$  is the so called Menger space, which will be denoted by  $(G, D, T)$ . If moreover  $G$  is a group, we say that  $(G, D, T)$  is a Menger group.

**Definition 7.** Let  $(G, D, \star)$  be a probabilistic metric space. A sequence  $(z_n) \subset G$  is said to be a Cauchy sequence if for all  $t \in \mathbb{R}^*$ ,

$$\lim_{n, p \rightarrow +\infty} D(z_n, z_p)(t) = \mathcal{H}_0(t).$$

A probabilistic metric space  $(G, D, \star)$  is said to be complete if every Cauchy sequence  $(z_n) \subset G$  converges to some  $z_\infty \in G$  i.e.  $\lim_{n \rightarrow +\infty} D(z_n, z_\infty)(t) = \mathcal{H}_0(t)$  for all  $t \in \mathbb{R}^*$ .

*Examples 2.* We give below two general frameworks of probabilistic invariant complete metric groups.

(1) Every (complete) metric space induce a probabilistic (complete) metric space. Indeed, if  $d$  is a metric on  $G$  and  $\star$  is a triangle function on  $\Delta^+$  satisfying  $\mathcal{H}_a \star \mathcal{H}_b = \mathcal{H}_{a+b}$  for all  $a, b \in \mathbb{R}^+$  (the triangle function  $\star_T$  with a left-continuous  $t$ -norm  $T$ , satisfies this condition, see for instance [8] and [15]), then  $(G, D, \star)$  where,

$$D(p, q) = \mathcal{H}_{d(p, q)}, \quad \forall p, q \in G,$$

is a probabilistic (complete) metric space. Thus, if  $(G, \cdot, d)$  is invariant complete metric group (Several examples of invariant complete metric groups are given in [3]) then  $(G, \cdot, D, \star)$  is a probabilistic invariant complete metric group.

(2) Let  $(G, F, \star)$  be a probabilistic complete normed vector space with a probabilistic norm  $F$  (the definition of probabilistic complete normed space can be found in [1], see also [8] p.66 for example). For all  $p, q \in G$ , let us set

$$D(p, q) = F_{p-q}.$$

Then,  $(G, +, D, \star)$  is a probabilistic invariant complete metric group (this example also works by replacing probabilistic normed vector space by probabilistic normed groups, the definition can be found in [20],[14]).

### 3. PROBABILISTIC 1-LIPSCHITZ MAP

In this section, we will begin by giving some basic properties of 1-Lipschitz maps, we then show that the space of all 1-Lipschitz maps is a probabilistic metric space. Finally, we give an analogue to the MC Shane result's on the extension of Lipschitz maps.

#### 3.1. Basic properties of 1-Lipschitz maps.

**Definition 8.** Let  $(G, D, \star)$  be a probabilistic metric space and let  $f : G \rightarrow \Delta^+$  be a map.

(1) We say that  $f$  is continuous at  $z \in G$  if  $\lim_{n \rightarrow +\infty} f(z_n)(t) = f(z)(t)$  for all  $t \in \mathbb{R}$  point of continuity of  $f(z)$ , whenever  $\lim_{n \rightarrow +\infty} D(z_n, z)(t) = \mathcal{H}_0(t)$  for all  $t \in \mathbb{R}^*$ .

(2) We say that  $f$  is 1-Lipschitz map if:

$$\forall x, y \in G, \quad D(x, y) \star f(y) \leq f(x).$$

*Remark 2.* We can also define  $k$ -Lipschitz maps  $f$  for a nonnegative real number  $k \geq 0$  as follows

$$\forall x, y \in G, \quad D_k(x, y) \star f(y) \leq f(x),$$

where, for all  $x, y \in G$  and all  $t \in \mathbb{R}$ ,  $D_k(x, y)(t) = D(x, y)(\frac{t}{k})$  if  $k > 0$  and  $D_0(x, y)(t) = \mathcal{H}_0(t)$  if  $k = 0$ .

*Examples 3.* Let  $(G, d)$  be a metric space and  $(G, D, \star)$  be the probabilistic metric space defined with the probabilistic metric

$$D(p, q) = \mathcal{H}_{d(p, q)}.$$

Assume that  $\star$  is a triangle function on  $\Delta^+$  satisfying  $\mathcal{H}_a \star \mathcal{H}_b = \mathcal{H}_{a+b}$  for all  $a, b \in \mathbb{R}^+$ . Let  $L : (G, d) \rightarrow \mathbb{R}^+$  be a classical nonnegative 1-Lipschitz map. Then, the map defined for all  $x \in G$  by

$$f(x) := \mathcal{H}_{L(x)}$$

is a probabilistic 1-Lipschitz map.

**Proposition 1.** *Every probabilistic 1-Lipschitz map is continuous.*

*Proof.* Let  $f$  be a probabilistic 1-Lipschitz map and  $(x_n)_n \subset G$  be a sequence such that  $\lim_n D(x_n, x) = \mathcal{H}_0$ . Let  $t \in \mathbb{R}$  a point of continuity of  $f(x)$ . On one hand, since  $(\Delta, \leq)$  is a complete lattice, the law  $\star$  is continuous and  $(D(x_n, x) \star f(x))(t) \leq f(x_n)(t)$ , then  $f(x)(t) = (\mathcal{H}_0 \star f(x))(t) = \liminf_n (D(x_n, x) \star f(x))(t) \leq \liminf_n f(x_n)(t)$ . Thus,  $f(x)(t) \leq \liminf_n f(x_n)(t)$ . On the other hand, since  $(D(x_n, x) \star f(x_n))(t) \leq f(x)(t)$ , it follows that  $\limsup_n f(x_n)(t) = (\mathcal{H}_0 \star \limsup_n f(x_n))(t) = \limsup_n ((D(x_n, x) \star f(x_n))(t)) \leq f(x)(t)$ . Thus,  $\lim_n f(x_n)(t) = f(x)(t)$  and so  $f$  is continuous.  $\square$

By  $Lip_\star^1(G, \Delta^+)$  we denote the space of all probabilistic 1-Lipschitz maps

$$Lip_\star^1(G, \Delta^+) := \{f : G \rightarrow \Delta^+ / D(x, y) \star f(y) \leq f(x); \forall x, y \in G\}.$$

For all  $x \in G$ , by  $\delta_x$  we denote the map

$$\begin{aligned} \delta_x : G &\longrightarrow \Delta^+ \\ y &\longmapsto D(y, x). \end{aligned}$$

We set  $\mathcal{G}(G) := \{\delta_x / x \in G\}$  and by  $\delta$ , we denote the operator

$$\begin{aligned} \delta : G &\longrightarrow \mathcal{G}(G) \subset Lip_\star^1(G, \Delta^+) \\ x &\longmapsto \delta_x. \end{aligned}$$

**Proposition 2.** *Let  $(G, D, \star)$  be a probabilistic metric space. Then, we have that  $\delta_a \in Lip_\star^1(G, \Delta^+)$  for each  $a \in G$  and the map  $\delta : G \rightarrow Lip_\star^1(G, \Delta^+)$  is injective.*

*Proof.* The fact that  $\delta_a \in Lip_\star^1(G, \Delta^+)$  for each  $a \in G$  follows from the property:  $D(x, y) \star D(y, a) \leq D(x, a)$  for all  $a, x, y \in G$ . Now, let  $a, b \in G$  be such that  $\delta_a = \delta_b$ . It follows that  $\delta_a(x) = \delta_b(x)$  for all  $x \in G$ . In particular, for  $x = b$  we have that  $D(a, b) = \delta_a(b) = \delta_b(b) = \mathcal{H}_0$ , which implies that

$a = b$ .

□

Let  $f \in Lip_\star^1(G, \Delta^+)$  and  $F \in \Delta^+$ , by  $\langle f, F \rangle : G \longrightarrow \Delta^+$ , we denote the map defined by  $\langle f, F \rangle(x) := f(x) \star F$  for all  $x \in G$ . We easily obtain the following propositions.

**Proposition 3.** *Let  $(G, D, \star)$  be a probabilistic metric space. Then, for all  $f \in Lip_\star^1(G, \Delta^+)$  and all  $F \in \Delta^+$ , we have that  $\langle f, F \rangle \in Lip_\star^1(G, \Delta^+)$ .*

**3.2. The probabilistic metric structure of the space  $Lip_\star^1(G, \Delta^+)$ .** We are going to build a probabilistic metric on the the space  $Lip_\star^1(G, \Delta^+)$ . Given a probabilistic metric space  $(G, D, \star)$ , we define a probabilistic metric  $\mathbb{D} : Lip_\star^1(G, \Delta^+) \times Lip_\star^1(G, \Delta^+) \longrightarrow \Delta^+$  as follows:

$$\mathbb{D}(f, g) = \begin{cases} \sup_{x \in G} (f(x) \star g(x)) & \text{if } f \neq g; (f, g) \in \mathcal{G}(G) \times \mathcal{G}(G) \\ \mathcal{H}_0 & \text{if } f = g \\ \inf_{x \in G} (f(x) \star g(x)) & \text{if } f \neq g; (f, g) \in [\mathcal{G}(G) \times \mathcal{G}(G)]^C \end{cases}$$

where, the set

$$[\mathcal{G}(G) \times \mathcal{G}(G)]^C := (Lip_\star^1(G, \Delta^+) \times Lip_\star^1(G, \Delta^+)) \setminus (\mathcal{G}(G) \times \mathcal{G}(G))$$

denotes the complement of  $\mathcal{G}(G) \times \mathcal{G}(G)$  in  $Lip_\star^1(G, \Delta^+) \times Lip_\star^1(G, \Delta^+)$ .

The interest of the probabilistic metric  $\mathbb{D}$  is that this allows to keep the distances between the points of  $G$  after embedding it in the space  $Lip_\star^1(G, \Delta^+)$  via the map  $\delta$ . This will play an important role in Theorem 5.

**Theorem 1.** *The map  $\mathbb{D}$  defines a probabilistic metric on  $Lip_\star^1(G, \Delta^+)$  satisfying for all  $a, b \in G$ ,*

$$\mathbb{D}(\delta_a, \delta_b) = D(a, b).$$

*Consequently, the map  $\delta : (G, D, \star) \longrightarrow (Lip_\star^1(G, \Delta^+), \mathbb{D}, \star)$  is an (probabilistic) isometry and so  $(G, D, \star)$  embeds isometrically into  $(Lip_\star^1(G, \Delta^+), \mathbb{D}, \star)$ .*

*Proof.* First, we show that, for all  $a, b \in G$ ,  $\mathbb{D}(\delta_a, \delta_b) = D(a, b)$ . Indeed, let  $a, b \in G$ . For all  $t \in \mathbb{R}$ , we have

$$\begin{aligned} \mathbb{D}(\delta_a, \delta_b) &= \sup_{x \in G} \delta_a(x) \star \delta_b(x) \\ &= \sup_{x \in G} D(x, a) \star D(x, b) \\ &\leq \sup_{x \in G} D(a, b) \text{ by property (iii) in Definition 4} \\ &= D(a, b). \end{aligned}$$



On the other hand,

$$\begin{aligned}
\mathbb{D}(\delta_a, \delta_b) &= \sup_{x \in G} \delta_a(x) \star \delta_b(x) \\
&\geq \delta_a(b) \star \delta_b(b) \\
&= D(a, b) \star \mathcal{H}_0 \\
&= D(a, b).
\end{aligned}$$

Thus,  $\mathbb{D}(\delta_a, \delta_b) = D(a, b)$ , for all  $a, b \in G$ .

Now, we show that  $\mathbb{D}$  is a probabilistic metric on  $Lip_\star^1(G, \Delta^+)$ .

(i) Let  $f, g \in Lip_\star^1(G, \Delta^+)$  such that  $f \neq g$ . We show that  $\mathbb{D}(f, g) \neq \mathcal{H}_0$ . Otherwise, suppose that  $\mathbb{D}(f, g) = \mathcal{H}_0$ . If  $(f, g) \in \mathcal{G}(G) \times \mathcal{G}(G)$ , there exists  $a \neq b \in G$  such that  $(f, g) = (\delta_a, \delta_b)$ . So,  $\mathbb{D}(f, g) = \mathbb{D}(\delta_a, \delta_b) = D(a, b) = \mathcal{H}_0$ , which is a contradiction since  $a \neq b$ . If  $(f, g) \in [\mathcal{G}(G) \times \mathcal{G}(G)]^C$ , then  $\mathbb{D}(f, g) = \inf_{x \in G} (f(x) \star g(x)) = \mathcal{H}_0$ . This implies that for all  $x \in G$ ,  $\mathcal{H}_0 \leq f(x) \star g(x) \leq f(x) \star \mathcal{H}_0 = f(x) \leq \mathcal{H}_0$  and so that  $f(x) = \mathcal{H}_0$  for all  $x \in G$ . In a similar way we prove that  $g(x) = \mathcal{H}_0$  for all  $x \in G$ . Thus  $f = g$  which is also a contradiction.

(ii) it is clear that  $\mathbb{D}(f, g) = \mathbb{D}(g, f)$  for all  $f, g \in Lip_\star^1(G, \Delta^+)$ .

(iii) let  $f, g, h \in Lip_\star^1(G, \Delta^+)$  and let us prove that  $\mathbb{D}(f, g) \star \mathbb{D}(g, h) \leq \mathbb{D}(f, h)$ . We treat three cases:

(a) if  $(f, g), (g, h) \in \mathcal{G}(G) \times \mathcal{G}(G)$ , there exists  $a, b, c \in G$  such that  $(f, g, h) = (\delta_a, \delta_b, \delta_c)$ . Thus,  $\mathbb{D}(f, g) \star \mathbb{D}(g, h) \leq \mathbb{D}(f, h) = D(a, b) \star D(b, c) \leq D(a, c) = \mathbb{D}(f, h)$ .

(b) if  $(f, g), (g, h) \in [\mathcal{G}(G) \times \mathcal{G}(G)]^C$ , then for all  $x \in G$  we have

$$\begin{aligned}
\mathbb{D}(f, g) \star \mathbb{D}(g, h) &\leq f(x) \star g(x) \star g(x), h(x) \\
&\leq f(x) \star \mathcal{H}_0 \star \mathcal{H}_0 \star h(x) \\
&= f(x) \star h(x)
\end{aligned}$$

Taking the infimum over  $x \in G$ , we get  $\mathbb{D}(f, g) \star \mathbb{D}(g, h) \leq \mathbb{D}(f, h)$ .

(c) if  $(f, g) \in \mathcal{G}(G) \times \mathcal{G}(G)$  and  $(g, h) \in [\mathcal{G}(G) \times \mathcal{G}(G)]^C$ , there exists  $a, b \in G$  such that  $(f, g) = (\delta_a, \delta_b)$  and  $h \notin \mathcal{G}(G)$ . Thus, for all  $x \in G$

$$\begin{aligned}
\mathbb{D}(f, g) \star \mathbb{D}(g, h) &= \mathbb{D}(\delta_a, \delta_b) \star \mathbb{D}(g, h) \\
&= D(a, b) \star \mathbb{D}(g, h) \\
&= D(a, b) \star \inf_{x \in G} (g(x) \star h(x)) \\
&\leq D(a, b) \star (g(x) \star h(x)) \\
&= (D(a, b) \star D(b, x)) \star h(x) \\
&\leq D(a, x) \star h(x) \\
&= f(x) \star h(x).
\end{aligned}$$

Thus, we get  $\mathbb{D}(f, g) \star \mathbb{D}(g, h) \leq \inf_{x \in G} (f(x) \star h(x)) = \mathbb{D}(f, h)$  by taking the infimum over  $x \in G$ .  $\square$

**3.3. The probabilistic analogue of Mac Shane result's.** Recall the Lipschitz extention result of Mac Shane in [10]: if  $(X, d)$  is a metric space,  $A$  a nonempty subset of  $X$  and  $f : A \rightarrow \mathbb{R}$  is  $k$ -Lipschitz map, then there exist a  $k$ -Lipschitz map  $\tilde{f} : X \rightarrow \mathbb{R}$  such that  $\tilde{f}|_A = f$ . We give bellow an analogue of this result for probabilistic 1-Lipschitz maps.

**Theorem 2.** *Let  $(G, D, \star)$  be a probabilistic metric space and  $A$  be a nonempty subset of  $G$ . Let  $f : A \rightarrow \Delta^+$  be a probabilistic 1-Lipschitz map. Then, there exists a probabilistic 1-Lipschitz map  $\tilde{f} : G \rightarrow \Delta^+$  such that  $\tilde{f}|_A = f$ .*

*Proof.* We define  $\tilde{f} : G \rightarrow \Delta^+$  as follows: for all  $x \in G$ ,

$$\tilde{f}(x) := \sup_{a \in A} D(a, x) \star f(a).$$

We first prove that  $\tilde{f}(x) = f(x)$  for all  $x \in A$ . Indeed, let  $x \in A$ . On one hand we have  $f(x) = \mathcal{H}_0 \star f(x) = D(x, x) \star f(x) \leq \sup_{a \in A} D(a, x) \star f(a) = \tilde{f}(x)$ . On the other hand, since  $f$  is probabilistic 1-Lipschitz on  $A$  and  $x \in A$ , then  $D(a, x) \star f(a) \leq f(x)$  for all  $a \in A$ . It follows that  $\tilde{f}(x) := \sup_{a \in A} D(a, x) \star f(a) \leq f(x)$ . Thus,  $\tilde{f}(x) = f(x)$  for all  $x \in A$ . Now, we show that  $\tilde{f}$  is probabilistic 1-Lipschitz on  $G$ . Indeed, Let  $x, y \in G$ . For all  $a \in A$  we have that  $D(a, x) \star D(x, y) \leq D(a, y)$ . So,  $D(a, x) \star f(a) \star D(x, y) \leq D(a, y) \star f(a)$ . By taking the supremum over  $a \in A$  and using axiom (vi) we get  $\tilde{f}(x) \star D(x, y) \leq \tilde{f}(y)$ . Hence,  $\tilde{f}$  is probabilistic 1-Lipschitz map on  $G$  that coincides with  $f$  on  $A$ .  $\square$

#### 4. PROBABILISTIC SUP-CONVOLUTION

In this section, we assume that  $G$  is a group. The aim of this section is to prove that if  $(G, \cdot, D, \star)$  is a probabilistic invariant metric group having  $e$  as identity element, then, the set  $(\text{Lip}_\star^1(G, \Delta^+), \mathbb{D}, \star)$  is a probabilistic metric monoid having  $\delta_e$  as identity element, when we equip it with a Sup-Convolution law defined below (Theorem 3). In this case, we prove that  $G$  embeds isomorphically as group into  $\text{Lip}_\star^1(G, \Delta^+)$  (Corollary 1). The space  $(\text{Lip}_\star^1(G, \Delta^+), \mathbb{D}, \odot)$  is an example of probabilistic metric monoid which is not a group.

**Definition 9.** *Let  $G$  be a group and let  $f, g : G \rightarrow \Delta^+$  be two maps. The Sup-Convolution of  $f$  and  $g$  denoted  $f \odot g : G \rightarrow \Delta^+$  is a map defined for all  $x \in G$  as follows*

$$\begin{aligned} (f \odot g)(x) &:= \sup_{y, z \in G / yz=x} \{(f(y) \star g(z))\} \\ &= \sup_{y \in G} \{(f(y) \star g(y^{-1}x))\}. \end{aligned}$$

We need to establish the three following propositions.

**Proposition 4.** *Let  $(G, \cdot, D, \star)$  be an invariant probabilistic metric group. Then, we have that  $\delta_a \odot \delta_b = \delta_{ab}$  for all  $a, b \in G$ .*

*Proof.* We first prove that  $\delta_e \odot \delta_e = \delta_e$ , where  $e$  is the identity element of  $G$ . Indeed, for all  $x \in G$  and all  $t \in \mathbb{R}$ ,

$$\begin{aligned}
 (\delta_e \odot \delta_e)(x) &:= \sup_{y, z \in G / yz=x} \{D(y, e) \star D(z, e)\} \\
 &= \sup_{y, z \in G / yz=x} \{(D(y, e) \star D(e, z^{-1}))\} \text{ by the invariance of } D \\
 &\leq \sup_{y, z \in G / yz=x} \{D(y, z^{-1})\} \text{ by the property (iii) of } D \\
 &= \sup_{y, z \in G / yz=x} \{D(yz, e)\} \text{ by the invariance of } D \\
 &= D(x, e). \\
 &= \delta_e(x)
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 (\delta_e \odot \delta_e)(x) &:= \sup_{y, z \in G / yz=x} \{(D(y, e) \star D(z, e))\} \\
 &\geq D(x, e) \star D(e, e) \\
 &= D(x, e) \star \mathcal{H}_0 \\
 &= D(x, e) \\
 &= \delta_e(x)
 \end{aligned}$$

Now, we prove that  $\delta_a \odot \delta_b = \delta_{ab}$  for all  $a, b \in G$ . Indeed, let  $x \in G$  and  $t \in \mathbb{R}$ , using the change variable  $u = zb^{-1}$  and the invariance of  $D$  we have:

$$\begin{aligned}
 \delta_a \odot \delta_b(x) &= \sup_{z \in X} \{(\delta_a(xz^{-1}) \star \delta_b(z))\} \\
 &= \sup_{z \in G} \{(\delta_e(xz^{-1}a^{-1}) \star \delta_e(zb^{-1}))\} \\
 &= \sup_{u \in G} \{(\delta_e(xb^{-1}t^{-1}a^{-1}) \star \delta_e(u))\} \\
 &= \sup_{u \in X} \{(\delta_e(a^{-1}xb^{-1}u^{-1}) \star \delta_e(u))\} \\
 &= (\delta_e \odot \delta_e)(a^{-1}xb^{-1}) \\
 &= \delta_e(a^{-1}xb^{-1}) \\
 &= \delta_e(xb^{-1}a^{-1}) \\
 &= \delta_e(x(ab)^{-1}) \\
 &= \delta_{ab}(x).
 \end{aligned}$$

Thus,  $\delta_a \odot \delta_b = \delta_{ab}$  for all  $a, b \in G$ . □

When  $G$  is a probabilistic invariant metric group, the space  $Lip_\star^1(G, \Delta^+)$  turns out to be naturally linked to the operation of Sup-Convolution. It is characterized by being the maximal set of maps from  $G$  into  $\Delta^+$  having  $\delta_e$  as identity element for the operation  $\odot$ .

**Proposition 5.** *Let  $(G, \cdot, D, \star)$  be an invariant probabilistic metric group. Then, we have*

$$Lip_\star^1(G, \Delta^+) = \{f : G \longrightarrow \Delta^+ \text{ map} / f \odot \delta_e = \delta_e \odot f = f\}.$$

*Proof.* Let  $f \in Lip_\star^1(G, \Delta^+)$ . We first prove that  $\delta_e \odot f(x) = f \odot \delta_e(x)$  for all  $x \in G$  and all  $t \in \mathbb{R}$ . Using the variable change  $u = xy^{-1}$  and the invariance of  $D$ , we have for all  $x \in G$  and for all  $t \in \mathbb{R}$ ,

$$\begin{aligned} (f \odot \delta_e)(x) &= \sup_{y \in G} \{ (f(xy^{-1}) \star \delta_e(y)) \} \\ &= \sup_{u \in G} \{ (f(u) \star \delta_e(u^{-1}x)) \} \\ &= \sup_{u \in G} \{ (\delta_e(xu^{-1}) \star f(u)) \} \\ &= (\delta_e \odot f)(x). \end{aligned}$$

Now, let  $f \in Lip_\star^1(G, \Delta^+)$  then for all  $x, y \in G$  and all  $t \in \mathbb{R}$  we have

$$\begin{aligned} f(x) &\geq (d(x, y) \star f(y)) \\ &= (d(xy^{-1}, e) \star f(y)) \\ &= (\delta_e(xy^{-1}) \star f(y)). \end{aligned}$$

Taking the supremum over  $y \in G$ , we get  $f(x) \geq (\delta_e \odot f)(x) = (f \odot \delta_e)(x)$  for all  $x \in G$  and all  $t \in \mathbb{R}$ . For the converse inequality we have

$$\begin{aligned} f \odot \delta_e(x) &= \sup_{y \in G} \{ f(xy^{-1}) \star \delta_e(y) \} \\ &\geq f(x) \star \delta_e(e) \\ &= f(x) \star \mathcal{H}_0 \\ &= f(x). \end{aligned}$$

Thus  $f \odot \delta_e = \delta_e \odot f = f$  for all  $f \in Lip_\star^1(G, \Delta^+)$  and so we have that  $Lip_\star^1(G, \Delta^+) \subset \{f : G \longrightarrow \Delta^+ \text{ map} / f \odot \delta_e = \delta_e \odot f = f\}$ . Now, let  $f \in \{f : G \longrightarrow \Delta^+ \text{ map} / f \odot \delta_e = \delta_e \odot f = f\}$ . It follows that for all  $x \in G$  and all  $t \in \mathbb{R}$

$$\begin{aligned} f(x) &= f \odot \delta_e(x) \\ &= \sup_{yz=x} \{ f(y) \star \delta_e(z) \} \\ &\geq f(y) \star \delta_e(y^{-1}x) \\ &= D(x, y) \star f(y). \end{aligned}$$

It follows that  $f \in Lip_\star^1(G, \Delta^+)$ . Finally we have  $Lip_\star^1(G, \Delta^+) = \{f : G \longrightarrow \Delta^+ \mid f \odot \delta_e = \delta_e \odot f = f\}$ .  $\square$

**Proposition 6.** *Let  $(G, \cdot, D, \star)$  be an invariant probabilistic metric group. Then, the set  $(Lip_\star^1(G, \Delta^+), \odot)$  has the associative property.*

*Proof.* Let  $f, g, h \in Lip_\star^1(G, \Delta^+)$ . Using the variable change  $u := yz$  together with the axiom (vi) after Definition 1, we obtain for all  $x \in G$ ,

$$\begin{aligned}
 (f \odot g) \odot h(x) &= \sup_{z \in G} \{(f \odot g)(xz^{-1}) \star h(z)\} \\
 &= \sup_{z \in G} \{(f \odot g)(xz^{-1}) \star h(z)\} \\
 &= \sup_{z \in G} \{\sup_{y \in G} \{f(xz^{-1}y^{-1}) \star g(y)\} \star h(z)\} \\
 &= \sup_{z \in G} \{\sup_{y \in G} \{f(xz^{-1}y^{-1}) \star g(y)\} \star h(z)\} \\
 &= \sup_{z \in G} \{\sup_{u \in G} \{f(xu^{-1}) \star g(uz^{-1})\} \star h(z)\} \\
 &= \sup_{u \in G} \{f(xu^{-1}) \star \sup_{z \in G} \{g(tz^{-1}) \star h(z)\}\} \\
 &= \sup_{u \in G} \{f(xu^{-1}) \star (g \odot h)(u)\} \\
 &= f \odot (g \odot h)(x).
 \end{aligned}$$

$\square$

**Theorem 3.** *Let  $(G, \cdot, D, \star)$  be an invariant probabilistic metric group. Then, the set  $(Lip_\star^1(G, \Delta^+), \odot, \mathbb{D}, \star)$  is a probabilistic metric monoid having  $\delta_e$  as identity element.*

*Proof.* We know that  $\mathbb{D}$  is a probabilistic metric on  $Lip_\star^1(G, \Delta^+)$ . Let us prove that  $(Lip_\star^1(G, \Delta^+), \odot)$  is a monoid having  $\delta_e$  as identity element. Indeed, by Proposition 5, if  $f, g \in Lip_\star^1(G, \Delta^+)$  then  $f \odot g \in Lip_\star^1(G, \Delta^+)$ . Indeed,  $\delta_e \odot f = f$ . Thus,  $(\delta_e \odot f) \odot g = f \odot g$ . By the associative property (Proposition 6), we have that  $\delta_e \odot (f \odot g) = f \odot g$ , which implies that  $f \odot g \in Lip_\star^1(G, \Delta^+)$ . This property with the associative property shows that  $(Lip_\star^1(G, \Delta^+), \odot)$  is a semigroup. Using Proposition 5, we obtain that  $(Lip_\star^1(G, \Delta^+), \odot)$  is a monoid having  $\delta_e$  as identity element.  $\square$

**Corollary 1.** *Let  $(G, \cdot, D, \star)$  be an probabilistic invariant metric group. Then, the map*

$$(2) \quad \delta : (G, \cdot, D, \star) \longrightarrow (Lip_\star^1(G, \Delta^+), \odot, \mathbb{D}, \star)$$

*is an injective probabilistic group morphism satisfying  $\mathbb{D}(\delta_a, \delta_b) = D(a, b)$  for all  $a, b \in G$ . In other words,  $(\mathcal{G}(G), \odot, \mathbb{D}, \star)$  is a probabilistic invariant metric group isometrically isomorphic to the probabilistic invariant metric group  $(G, \cdot, D, \star)$ .*

*Proof.* The proof is a consequence of Theorem 1 and Proposition 4.  $\square$

## 5. THE MAIN RESULTS

Let  $\mathcal{U}(\Delta^+)$  (resp.  $\mathcal{U}(Lip_\star^1(G, \Delta^+))$ ) denotes the group of all invertible element of  $(\Delta^+, \star)$  (resp. of  $(Lip_\star^1(G, \Delta^+), \odot)$ ). Recall that if  $f \in Lip_\star^1(G, \Delta^+)$  and  $F \in \Delta^+$ , then  $\langle f, F \rangle(x) := f(x) \star F$  for all  $x \in G$ . Recall also that  $\mathcal{G}(G) := \{\delta_a/a \in G\}$ . By  $\langle \mathcal{G}(G), \mathcal{U}(\Delta^+) \rangle$ , we denote the following subgroup of  $(Lip_\star^1(G, \Delta^+), \odot)$

$$\langle \mathcal{G}(G), \mathcal{U}(\Delta^+) \rangle := \{\langle \delta_a, U \rangle / a \in G \text{ and } U \in \mathcal{U}(\Delta^+)\}.$$

Note that if  $\mathcal{U}(\Delta^+)$  is trivial i.e.  $\mathcal{U}(\Delta^+) = \{\mathcal{H}_0\}$ , then  $\langle \mathcal{G}(G), \mathcal{U}(\Delta^+) \rangle = \mathcal{G}(G)$ . The symbol " $\cong$ " means "isometrically isomorphic".

*Remark 3.* Let  $\star$  be a triangle function on  $\Delta^+$  such that  $\mathcal{U}(\Delta^+) = \{\mathcal{H}_0\}$  (see Lemma 1). We equip  $\Delta^+$  with the following probabilistic metric:

$$D(K, L) = \begin{cases} \mathcal{H}_0 & \text{if } K = L \\ K \star L & \text{otherwise} \end{cases}$$

Then,  $(\Delta^+, \star, D) \cong (Lip_\star^1(\{\mathcal{H}_0\}, \Delta^+), \odot, \mathbb{D})$ .

**Theorem 4.** Let  $(G, \cdot, D, \star)$  be an invariant probabilistic complete metric group. Then, we have,

$$\mathcal{U}(Lip_\star^1(G, \Delta^+)) = \langle \mathcal{G}(G), \mathcal{U}(\Delta^+) \rangle.$$

Moreover,

- (1) for all  $a \in G$  and  $U \in (\mathcal{U}(\Delta^+), \star)$ , we have  $\langle \delta_a, U \rangle^{-1} = \langle \delta_{a^{-1}}, U^{-1} \rangle$ .
- (2) if  $\mathcal{U}(\Delta^+) = \{\mathcal{H}_0\}$ , then

$$\begin{aligned} (\mathcal{U}(Lip_\star^1(G, \Delta^+)), \odot, \mathbb{D}, \star) &= (\mathcal{G}(G), \odot, \mathbb{D}, \star) \\ &\cong (G, \cdot, D, \star). \end{aligned}$$

*Proof.* We first prove that  $\{\langle \delta_a, U \rangle / a \in G, U \in \mathcal{U}(\Delta^+)\} \subset \mathcal{U}(Lip_\star^1(G, \Delta^+))$ . Indeed, using the definition of the operation  $\odot$  and the axiom (vi) after Definition 1, it is easy to see that  $\langle \delta_a, U \rangle \odot \langle \delta_b, V \rangle = \langle \delta_a \odot \delta_b, U \star V \rangle$  for all  $a, b \in G$  and all  $U, V \in \mathcal{U}(\Delta^+)$ . Using Proposition 4, we get  $\langle \delta_a, U \rangle \odot \langle \delta_{a^{-1}}, U^{-1} \rangle = \langle \delta_e, \mathcal{H}_0 \rangle = \delta_e$  for all  $a \in G$  and all  $U \in \mathcal{U}(\Delta^+)$ . Hence,  $\langle \delta_a, U \rangle \in Lip_\star^1(G, \Delta^+)$  and  $\langle \delta_a, U \rangle^{-1} = \langle \delta_{a^{-1}}, U^{-1} \rangle$ .

Now, we are going to prove the converse. Indeed, let  $f, g \in Lip_\star^1(G, \Delta^+)$  be such that

$$(3) \quad f \odot g = \delta_e$$

In particular, we have  $f \odot g(e) = \delta_e(e) = \mathcal{H}_0$ . Thus, we obtain

$$\begin{cases} (f \odot g)(e)(t) = 0, & \text{for all } t \leq 0 \\ (f \odot g)(e)(t) = 1, & \text{for all } t > 0 \end{cases}$$

So, on one hand, we have that for all  $z \in G$  and for all  $t \leq 0$

$$(4) \quad (f(z) \star g(z^{-1}))(t) = 0$$

On the other hand, we have that for all  $t > 0$

$$(5) \quad \sup_{z \in G} \{(f(z) \star g(z^{-1}))(t)\} = 1$$

For all  $n \in \mathbb{N}$ , by taking  $t = \frac{1}{n}$  in (5), there exists  $(z_n) \subset G$  such that for all  $n \in \mathbb{N}^*$

$$1 - \frac{1}{n} < (f(z_n) \star g(z_n^{-1}))(\frac{1}{n}) \leq 1$$

Since  $f(z_n) \star g(z_n^{-1}) \in \Delta^+$  is increasing then, for all  $t > 0$  there exists  $N_t \in \mathbb{N}$  such that for all  $n \geq N_t$  we have

$$(6) \quad 1 - \frac{1}{n} < (f(z_n) \star g(z_n^{-1}))(t) \leq 1$$

From (4) and (6) we have that for all  $t \in \mathbb{R}$

$$(7) \quad \lim_{n \rightarrow +\infty} (f(z_n) \star g(z_n^{-1}))(t) = \mathcal{H}_0(t)$$

From the definition of  $f \odot g$ , we get that for all  $x, y \in G$

$$(8) \quad f(x) \star g(y^{-1}) \leq (f \odot g)(xy^{-1})$$

Using (v) of Definition 1 and (8), we get that

$$(f(z_n) \star g(z_p^{-1})) \star (f(z_p) \star g(z_n^{-1})) \leq (f \odot g)(z_n z_p^{-1}) \star (f \odot g)(z_p z_n^{-1})$$

By the commutativity of the operation  $\star$ , the invariance of  $D$  and (3), we obtain

$$(9) \quad \begin{aligned} (f(z_n) \star g(z_n^{-1})) \star (f(z_p) \star g(z_p^{-1})) &\leq \delta_e(z_n z_p^{-1}) \star \delta_e(z_p z_n^{-1}) \\ &= D(z_n, z_p) \star D(z_n, z_p). \end{aligned}$$

Using the fact that  $\mathcal{H}_0$  is the maximal element of  $\Delta^+$ , we get that for all  $t \in \mathbb{R}$  and all  $n, p \in \mathbb{N}^*$

$$(10) \quad \begin{aligned} D(z_n, z_p) \star D(z_n, z_p)(t) &\leq (D(z_n, z_p) \star \mathcal{H}_0)(t) \\ &= D(z_n, z_p)(t) \\ &\leq \mathcal{H}_0(t). \end{aligned}$$

Using (7), (9), (10) and the continuity of  $\star$ , we have that, for all  $t \in \mathbb{R}^*$

$$\lim_{n, p \rightarrow +\infty} D(z_n, z_p)(t) = \mathcal{H}_0(t).$$

Since  $(G, D, \star)$  is a probabilistic complete metric group, the sequence  $(z_n)$  converges to some  $z_\infty \in G$  i.e.  $\lim_{n \rightarrow +\infty} D(z_n, z_\infty)(t) = \mathcal{H}_0(t)$  for all  $t \in \mathbb{R}$ .

Also, we have that  $(z_n^{-1})$  converges to  $z_\infty^{-1}$  by the invariance of  $D$ . Using the continuity of  $f, g \in Lip_\star^1(G, \Delta^+)$ , the continuity of the law  $\star$  in the formula (7), we get that  $f(z_\infty) \star g(z_\infty^{-1}) = \mathcal{H}_0$ . So,  $f(z_\infty), g(z_\infty^{-1}) \in (\mathcal{U}(\Delta^+), \star)$  and  $g(z_\infty^{-1}) = (f(z_\infty))^{-1}$ . Using the definition of  $f \odot g$  and the fact that  $f \odot g = \delta_e$ , we have for all  $x \in G$

$$f(x) \star g(z_\infty^{-1}) \leq f \odot g(xz_\infty^{-1}) = \delta_e(xz_\infty^{-1}) = D(x, z_\infty)$$

Hence from Definition 1 (v), we get that

$$f(x) \leq D(x, z_\infty) \star f(z_\infty).$$

On the other hand, since  $f \in Lip_\star^1(G, \Delta^+)$ , then  $f(x) \leq D(x, z_\infty) \star f(z_\infty)$  and so we obtain that  $f(x) = D(x, z_\infty) \star f(z_\infty) =: \langle \delta_{z_\infty}, f(z_\infty) \rangle(x)$  for all  $x \in G$ . Thus,  $f = \langle \delta_{z_\infty}, f(z_\infty) \rangle$  with  $z_\infty \in G$  and  $f(z_\infty) \in (\mathcal{U}(\Delta^+), \star)$ . This finish the proof.  $\square$

We obtain the following probabilistic Banach-Stone type theorem. Note that a general framework where the group of invertible elements of  $(\Delta^+, \star)$  is trivial (i.e.  $\mathcal{U}(\Delta^+) = \{\mathcal{H}_0\}$ ) is given in Lemma 1 below. The following theorem shows that the structure of the probabilistic invariant complete metric group  $(G, \cdot, D, \star)$  such that the group of invertible elements of  $(\Delta^+, \star)$  is trivial, is completely determined by the probabilistic metric monoid structure of  $Lip_\star^1(G, \Delta^+)$ . This is the case in particular for every invariant complete Menger groups (Corollary 2).

**Theorem 5.** *Let  $(G, \cdot, D, \star)$  and  $(G', \cdot, D', \star)$  be two probabilistic invariant complete metric groups such that the group of invertible elements of  $(\Delta^+, \star)$  is trivial. Then, the following assertions are equivalent.*

- (1)  *$(G, \cdot, D, \star)$  and  $(G', \cdot, D', \star)$  are isometrically isomorphic as probabilistic groups,*
- (2)  *$(Lip_\star^1(G, \Delta^+), \mathbb{D}, \odot)$  and  $(Lip_\star^1(G', \Delta^+), \mathbb{D}, \odot)$  are isometrically isomorphic as probabilistic monoids.*

*Proof.* (1)  $\implies$  (2). Suppose that  $\mathcal{I} : G \longrightarrow G'$  is a group isomorphism such that, for all  $x, y \in G$   $D'(\mathcal{I}(x), \mathcal{I}(y)) = D(x, y)$ . Then, it is easy to see that the map defined by  $\Phi(f) := f \circ \mathcal{I}^{-1}$  for all  $f \in Lip_\star^1(G, \Delta^+)$  maps the space  $Lip_\star^1(G, \Delta^+)$  into the space  $Lip_\star^1(G', \Delta^+)$  and satisfies  $\Phi(f \odot g) = \Phi(f) \odot \Phi(g)$  for all  $f, g \in Lip_\star^1(G, \Delta^+)$  so  $\Phi$  is a monoid isomorphism. Also we can easily see that  $\mathbb{D}(\Phi(f), \Phi(g)) = \mathbb{D}(f, g)$  for all  $f, g \in Lip_\star^1(G, \Delta^+)$ , so  $\Phi$  is an isometry.

(2)  $\implies$  (1). Suppose that  $\Phi : Lip_\star^1(G, \Delta^+) \longrightarrow Lip_\star^1(G', \Delta^+)$  is an isomorphism of monoids such that  $\mathbb{D}(\Phi(f), \Phi(g)) = \mathbb{D}(f, g)$  for all  $f, g \in Lip_\star^1(G, \Delta^+)$ . Since an isomorphism of monoids send the group of invertibles elements on the group of invertible elements, then by using Theorem 4, we obtain that the restriction  $\Phi|_{\langle \mathcal{G}(G), \mathcal{U}(\Delta^+) \rangle} : \langle \mathcal{G}(G), \mathcal{U}(\Delta^+) \rangle \longrightarrow \langle \mathcal{G}(G'), \mathcal{U}(\Delta^+) \rangle$  of



$\Phi$  is an isometric isomorphism. Since  $\mathcal{U}(\Delta^+) = \{\mathcal{H}_0\}$ , then  $\langle \mathcal{G}(G), \mathcal{U}(\Delta^+) = \mathcal{G}(G)$  and so  $\Phi|_{\mathcal{G}(G)} : \mathcal{G}(G) \rightarrow \mathcal{G}(G')$  is an isometric isomorphism. Using Corollary 1, we obtain that  $\mathcal{I} := \delta^{-1} \circ \Phi|_{\mathcal{G}(G)} \circ \delta : G \rightarrow G'$  is an isomorphism and satisfies  $D'(\mathcal{I}(a), \mathcal{I}(b)) = D(a, b)$  for all  $a, b \in G$ .  $\square$

In the following lemma, we prove that the set of all invertible element of  $(\Delta^+, \star)$  is trivial (i.e.  $\mathcal{U}(\Delta^+) = \{\mathcal{H}_0\}$ ) in the case where the law  $\star = \star_T$  is defined for all  $L, K \in \Delta^+$  and for all  $t \in \mathbb{R}$  by:

$$(11) \quad (L \star_T K)(t) := \sup_{s+u=t} T(L(s), K(u))$$

where,  $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$  is a left-continuous  $t$ -norm.

**Lemma 1.** *The distribution  $\mathcal{H}_0$  is the only one invertible element of the monoid  $(\Delta^+, \star_T)$ .*

*Proof.* Let  $L, K \in \Delta^+$  be such that  $L \star_T K = \mathcal{H}_0$ . Then, for all  $t > 0$ ,

$$\sup_{s+u=t} T(L(s), K(u)) = 1.$$

In particular, for each  $n \in \mathbb{N}^*$ ,

$$(12) \quad \sup_{s+u=\frac{1}{n}} T(L(s), K(u)) = 1$$

Thus, from (12), there exists a sequence  $(s_n)$  of real numbers such that

$$1 - \frac{1}{n} < T(L(s_n), K(\frac{1}{n} - s_n)).$$

Using the monotonicity of  $T$  and the fact that  $T(x, y) \leq T(1, y) = y \leq 1$  for all  $x, y \in [0, 1]$ , we obtain for all  $t > 0$ , there exists  $N_t \in \mathbb{N}$  such that for all  $n \geq N_t$  we have

$$1 - \frac{1}{n} < T(L(s_n), K(\frac{1}{n} - s_n)) \leq T(L(s_n), K(t - s_n)) \leq K(t - s_n) \leq 1,$$

Since  $T(0, x) = T(x, 0) = 0$  for all  $x \in [0, 1]$ , we deduce from the above inequality that necessarily  $0 < s_n < \frac{1}{n}$  for all  $n \in \mathbb{N}^*$  (since,  $L(s_n)$  and  $K(\frac{1}{n} - s_n)$  must be different from 0). Thus,  $s_n \searrow 0$  and for all  $t > 0$ , there exists  $N_t \in \mathbb{N}$  such that for all  $n \geq N_t$

$$1 - \frac{1}{n} < K(t - s_n) \leq 1,$$

Since  $K$  is left-continuous, we get that  $K(t) = 1$  for all  $t > 0$ . On the other hand, we know that  $K(t) = 0$  for all  $t \leq 0$ . It follows that  $K = \mathcal{H}_0$  and so also  $L = \mathcal{H}_0$ . In consequence, we have  $\mathcal{U}(\Delta^+) = \{\mathcal{H}_0\}$ .  $\square$

We obtain in the following corollary the result announced in the abstract. In other words, let  $T$  be a left-continuous  $t$ -norm then, every probabilistic invariant complete Menger group  $(G, \cdot, D, T)$  is completely determined by the probabilistic metric monoid structure of  $(Lip_\star^1(G, \Delta^+), \mathbb{D}, \odot)$ .

**Corollary 2.** *Let  $T$  be a left-continuous  $t$ -norm and  $(G, \cdot, D, T)$ ,  $(G', \cdot, D', T)$  be two probabilistic invariant complete Menger groups. Then,  $(G, \cdot, D, T)$  and  $(G', \cdot, D', T)$  are isometrically isomorphic as Menger groups if and only if  $(Lip_\star^1(G, \Delta^+), \odot, \mathbb{D}, \star)$  and  $(Lip_\star^1(G', \Delta^+), \odot, \mathbb{D}, \star)$  are isometrically isomorphic as probabilistic metric monoids.*

*Proof.* The proof is a direct consequence of Theorem 5 and Lemma 1.  $\square$

Also, we have that  $\mathcal{U}(\Delta^+) = \{\mathcal{H}_0\}$  if  $\star = \star_{T^*}$  is defined for all  $L, K \in \Delta^+$  and for all  $t \in \mathbb{R}$  by:

$$(13) \quad (L \star_{T^*} K)(t) := \inf_{s+u=t} T^*(L(s), K(u))$$

where,  $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$  is a  $t$ -norm and  $T^*(x, y) := 1 - T(1 - x, 1 - y)$  for all  $x, y \in [0, 1]$  ( $T^*$  is a dual function of  $T$  called  $t$ -conorm).

**Proposition 7.** *The distribution  $\mathcal{H}_0$  is the only one invertible element of the monoid  $(\Delta^+, \star_{T^*})$ .*

*Proof.* Let  $L, K \in \Delta^+$  be such that  $L \star_{T^*} K = \mathcal{H}_0$ . Then, for all  $t \in \mathbb{R}$ ,

$$\inf_{s+u=t} T^*(L(s), K(u)) = \mathcal{H}_0(t).$$

In other words, for all  $t \in \mathbb{R}$

$$\sup_{s+u=t} T(1 - L(s), 1 - K(u)) = 1 - \mathcal{H}_0(t).$$

In particular, for each  $n \in \mathbb{N}^*$ ,

$$(14) \quad \sup_{s+u=\frac{1}{n}} T(1 - L(s), 1 - K(u)) = 0$$

Thus, from (14), for all  $n \in \mathbb{N}^*$ ,

$$T(1 - L(\frac{1}{n}), 1 - K(0)) = 0.$$

Since  $K(0) = 0$ , using the fact that  $T(x, 1) = x$  for all  $x \in [0, 1]$ , we obtain that  $L(\frac{1}{n}) = 1$  for all  $n \in \mathbb{N}^*$ . Since  $L$  is nondecreasing, we get that for all  $t > 0$ ,  $L(t) = 1$ . So that  $L = \mathcal{H}_0$  and so also  $K = \mathcal{H}_0$ .  $\square$

*Remark 4.* Note that  $\star_{T^*}$  is not sup-continuous (i.e. does not verify the axiom (vi) after Definition 1) and therefore not used in this paper.

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LABORATOIRE SAMM 4543, UNIVERSITÉ PARIS 1 PANTHÉON-SORBONNE, CENTRE P.M.F.  
 90 RUE TOLBIAC, 75634 PARIS CEDEX 13, FRANCE  
*E-mail address:* Mohammed.Bachir@univ-paris1.fr