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INTRODUCTION TO THE PROBABILISTIC 1-LIPSCHITZ MAPS

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ABSTRACT. We introduce the notion of probabilistic 1-Lipschitz map defined on a probabilistic metric space and give an analogous to the result of Mac Shane on the extension of real-valued Lipschitz functions from a subset of a metric space to the whole space.

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1. INTRODUCTION

We introduce the notion of probabilistic 1-Lipschitz map defined on a probabilistic metric space and give an analogous to the result of Mac Shane (see [2]) on the extension of real-valued Lipschitz functions from a subset of a metric space to the whole space.

A distribution function is a function $F : [-\infty, +\infty] \rightarrow [0, 1]$ which is non-decreasing and left-continuous with $F(-\infty) = 0$; $F(+\infty) = 1$. The set of all distribution functions will be denoted by Δ . The subset of Δ consisting on distributions F such that $F(0) = 0$ will be denoted by Δ^+ . For $F, G \in \Delta^+$, the relation $F \leq G$ is meant by $F(t) \leq G(t)$, for all $t \in \mathbb{R}$. For all $a \in \mathbb{R}$, the distribution H_a is defined as follow

$$H_a(t) = \begin{cases} 0, & \text{if } t \leq a \\ 1, & \text{if } t > a \end{cases}$$

For $a = +\infty$,

$$H_\infty(t) = \begin{cases} 0, & \text{if } t \in [-\infty, +\infty[\\ 1, & \text{if } t = +\infty \end{cases}$$

It is well known that (Δ, \leq) and (Δ^+, \leq) are complete lattices with the minimal element H_∞ and the maximal element H_0 (see [1]). Thus, for any nonempty set I and any family $(F_i)_{i \in I}$ of distributions in Δ (resp. in Δ^+), the function $F = \sup_{i \in I} F_i$ is also an element of Δ (resp. of Δ^+).

Axioms 1.1. *In this work, we assume that Δ^+ is equipped with a law \star (a triangular function) satisfying the following axioms:*

- (i) $F \star L \in \Delta^+$ for all $F, L \in \Delta^+$.
- (ii) $F \star L = L \star F$ for all $F, L \in \Delta^+$.
- (iii) $F \star (L \star K) = (F \star L) \star K$, for all $F, L, K \in \Delta^+$.
- (iv) $F \star H_0 = F$ for all $F \in \Delta^+$.
- (v) $F \leq L \implies F \star K \leq L \star K$ for all $F, L, K \in \Delta^+$.
- (vi) Let I be a set, $(F_i)_{i \in I}$ a family of distributions in Δ^+ and $L \in \Delta^+$. Then, $\sup_{i \in I} (F_i \star L) = (\sup_{i \in I} F_i) \star L$.

Definition 1.2. *We say that \star is continuous at $(F, L) \in \Delta^+ \times \Delta^+$ if $\lim_{n \rightarrow +\infty} (F_n \star L_n)(t) = (F \star L)(t)$ for all $t \in \mathbb{R}$ point of continuity of $F \star L$, whenever $\lim_{n \rightarrow +\infty} F_n(t) = F(t)$ for all $t \in \mathbb{R}$ point of continuity of F and $\lim_{n \rightarrow +\infty} L_n(t) = L(t)$ for all $t \in \mathbb{R}$ point of continuity of L .*

Example 1.3. (see [3], [1]) *Let $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be a left-continuous t -norm, then the operation \star defined for all $F, L \in \Delta^+$ and for all $t \in \mathbb{R}$ by*

$$(F \star L)(t) := \sup_{s+u=t} T(F(s), L(u)) \quad (1.1)$$

is continuous at each point $(F, L) \in \Delta^+ \times \Delta^+$ and satisfies the axioms (i)-(vi).

In all this work, we assume that Δ^+ is equipped with a continuous law \star satisfying the axioms (i)-(vi).

Definition 1.4. *Let G be a set and let $D : G \times G \rightarrow (\Delta^+, \star, \leq)$ be a map. We say that (G, D, \star) is a probabilistic metric space if the following axioms (i)-(iii) hold:*

- (i) $D(p, q) = H_0$ iff $p = q$.
- (ii) $D(p, q) = D(q, p)$ for all $p, q \in G$
- (iii) $D(p, q) \star D(q, r) \leq D(p, r)$ for all $p, q, r \in G$

2. PROBABILISTIC 1-LIPSCHITZ MAPS

We define probabilistic continuous functions.

Definition 2.1. *Let (G, D, \star) be a probabilistic metric space and $f : G \rightarrow \Delta$ be a function. We say that f is continuous at $x \in G$ if $\lim_n f(x_n)(t) = f(x)(t)$ for all $t \in \mathbb{R}$ point of continuity of $f(x)$ and all sequence $(x_n)_n \subset G$ such that $\lim_n D(x_n, x) = H_0$ (i.e. such that $\lim_n D(x_n, x)(t) = H_0(t)$ for all $t \in \mathbb{R}$).*

Now, we introduce the notion of probabilistic 1-Lipschitz map.

Definition 2.2. *Let (G, D, \star) be a probabilistic metric space and $f : G \rightarrow \Delta$ be a map. We say that f is a probabilistic 1-Lipschitz map if, for all $x, y \in G$ we have:*

$$D(x, y) \star f(y) \leq f(x).$$

Proposition 2.3. *Every probabilistic 1-Lipschitz map is continuous.*

Proof. Let f be a probabilistic 1-Lipschitz map and $(x_n)_n \subset G$ be a sequence such that $\lim_n D(x_n, x) = H_0$. On one hand, since (Δ, \leq) is a complete lattice, the law \star is continuous and $D(x_n, x) \star f(x) \leq f(x_n)$, then $f(x) = H_0 \star f(x) = \liminf_n D(x_n, x) \star f(x) \leq \liminf_n f(x_n)$. Thus, $f(x) \leq \liminf_n f(x_n)$. On the other hand, since $D(x_n, x) \star f(x_n) \leq f(x)$, it follows that $\limsup_n f(x_n) = H_0 \star \limsup_n f(x_n) = \limsup_n (D(x_n, x) \star f(x_n)) \leq f(x)$. Thus, $\lim_n f(x_n) = f(x)$ and so f is continuous. \square

By $Lip_\star^1(G, \Delta)$ (resp. $Lip_\star^1(G, \Delta^+)$), we denotes the space of all probabilistic 1-Lipschitz maps (resp. all Δ^+ -valued 1-Lipschitz maps). For all $x \in G$, by $\delta_x : G \rightarrow \Delta^+$ we denote the map $\delta_x : y \mapsto D(y, x)$ and by δ , we denote the operator $\delta : x \mapsto \delta_x$.

Proposition 2.4. *Let (G, D, \star) be a probabilistic metric space. Then, we have that $\delta_a \in Lip_\star^1(G, \Delta^+)$ for each $a \in G$ and the map $\delta : G \rightarrow Lip_\star^1(G, \Delta^+)$ is injective.*

Proof. The fact that $\delta_a \in Lip_\star^1(G, \Delta^+)$ for each $a \in G$ follows from the property: $D(x, y) \star D(y, a) \leq D(x, a)$ for all $a, x, y \in G$. Now, let $a, b \in G$ be such that $\delta_a = \delta_b$. It follows that $\delta_a(x) = \delta_b(x)$ for all $x \in G$. In particular, for $x = b$ we have that $D(a, b) = \delta_a(b) = \delta_b(b) = H_0$, which implies that $a = b$. \square

Let $f \in Lip_\star^1(G, \Delta^+)$ and $F \in \Delta^+$, by $\langle f, F \rangle : G \rightarrow \Delta^+$, we denote the map defined by $\langle f, F \rangle(x) := f(x) \star F$ for all $x \in G$. We easily obtain the following proposition.

Proposition 2.5. *Let (G, D, \star) be a probabilistic metric space. Then, for all $f \in Lip_\star^1(G, \Delta^+)$ and all $F \in \Delta^+$, we have that $\langle f, F \rangle \in Lip_\star^1(G, \Delta^+)$.*

Recall the Lipschitz extension result of Mac Shane in [2]: if (X, d) is a metric space, A a nonempty subset of X and $f : A \rightarrow \mathbb{R}$ is k -Lipschitz map, then there exist a k -Lipschitz map $\tilde{f} : X \rightarrow \mathbb{R}$ such that $\tilde{f}|_A = f$. We give bellow an analogous of this result for probabilistic 1-Lipschitz maps.

Theorem 2.6. *Let (G, D, \star) be a probabilistic metric space and A be a nonempty subset of G . Let $f : A \rightarrow \Delta$ be a probabilistic 1-Lipschitz map. Then, there exists a probabilistic 1-Lipschitz map $\tilde{f} : G \rightarrow \Delta$ such that $\tilde{f}|_A = f$.*

Proof. We define $\tilde{f} : G \rightarrow \Delta$ as follows: for all $x \in G$,

$$\tilde{f}(x) := \sup_{a \in A} D(a, x) \star f(a).$$

We first prove that $\tilde{f}(x) = f(x)$ for all $x \in A$. Indeed, let $x \in A$. On one hand we have $f(x) = H_0 \star f(x) = D(x, x) \star f(x) \leq \sup_{a \in A} D(a, x) \star f(a) = \tilde{f}(x)$. On the other hand, since f is probabilistic 1-Lipschitz on A and $x \in A$, then $D(a, x) \star f(a) \leq f(x)$ for all $a \in A$. It follows that $\tilde{f}(x) := \sup_{a \in A} D(a, x) \star f(a) \leq f(x)$. Thus, $\tilde{f}(x) = f(x)$ for all $x \in A$. Now, we show that \tilde{f} is probabilistic 1-Lipschitz on G . Indeed, Let $x, y \in G$. For all $a \in A$ we have that $D(a, x) \star D(x, y) \leq D(a, y)$. So, $D(a, x) \star f(a) \star D(x, y) \leq D(a, y) \star f(a)$. By taking the supremum over $a \in A$ and using axiom (vi) we get $\tilde{f}(x) \star D(x, y) \leq \tilde{f}(y)$. Hence, \tilde{f} is probabilistic 1-Lipschitz map on G that coincides with f on A . \square

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