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# INTRODUCTION TO THE PROBABILISTIC 1-LIPSCHITZ MAPS

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ABSTRACT. We introduce the notion of probabilistic 1-Lipschitz map defined on a probabilistic metric space and give an analogous to the result of Mac Shane on the extension of real-valued Lipschitz functions from a subset of a metric space to the whole space.

54E70, 47S50, 46S50

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## 1. INTRODUCTION

We introduce the notion of probabilistic 1-Lipschitz map defined on a probabilistic metric space and give an analogous to the result of Mac Shane (see [2]) on the extension of real-valued Lipschitz functions from a subset of a metric space to the whole space.

A distribution function is a function  $F : [-\infty, +\infty] \rightarrow [0, 1]$  which is non-decreasing and left-continuous with  $F(-\infty) = 0$ ;  $F(+\infty) = 1$ . The set of all distribution functions will be denoted by  $\Delta$ . The subset of  $\Delta$  consisting on distributions  $F$  such that  $F(0) = 0$  will be denoted by  $\Delta^+$ . For  $F, G \in \Delta^+$ , the relation  $F \leq G$  is meant by  $F(t) \leq G(t)$ , for all  $t \in \mathbb{R}$ . For all  $a \in \mathbb{R}$ , the distribution  $H_a$  is defined as follow

$$H_a(t) = \begin{cases} 0, & \text{if } t \leq a \\ 1, & \text{if } t > a \end{cases}$$

For  $a = +\infty$ ,

$$H_\infty(t) = \begin{cases} 0, & \text{if } t \in [-\infty, +\infty[ \\ 1, & \text{if } t = +\infty \end{cases}$$

It is well known that  $(\Delta, \leq)$  and  $(\Delta^+, \leq)$  are complete lattices with the minimal element  $H_\infty$  and the maximal element  $H_0$  (see [1]). Thus, for any nonempty set  $I$  and any family  $(F_i)_{i \in I}$  of distributions in  $\Delta$  (resp. in  $\Delta^+$ ), the function  $F = \sup_{i \in I} F_i$  is also an element of  $\Delta$  (resp. of  $\Delta^+$ ).

**Axioms 1.1.** *In this work, we assume that  $\Delta^+$  is equipped with a law  $\star$  (a triangular function) satisfying the following axioms:*

- (i)  $F \star L \in \Delta^+$  for all  $F, L \in \Delta^+$ .
- (ii)  $F \star L = L \star F$  for all  $F, L \in \Delta^+$ .
- (iii)  $F \star (L \star K) = (F \star L) \star K$ , for all  $F, L, K \in \Delta^+$ .
- (iv)  $F \star H_0 = F$  for all  $F \in \Delta^+$ .
- (v)  $F \leq L \implies F \star K \leq L \star K$  for all  $F, L, K \in \Delta^+$ .

(vi) Let  $I$  be a set,  $(F_i)_{i \in I}$  a family of distributions in  $\Delta^+$  and  $L \in \Delta^+$ . Then,  $\sup_{i \in I} (F_i \star L) = (\sup_{i \in I} F_i) \star L$ .

**Definition 1.2.** *We say that  $\star$  is continuous at  $(F, L) \in \Delta^+ \times \Delta^+$  if  $\lim_{n \rightarrow +\infty} (F_n \star L_n)(t) = (F \star L)(t)$  for all  $t \in \mathbb{R}$  point of continuity of  $F \star L$ , whenever  $\lim_{n \rightarrow +\infty} F_n(t) = F(t)$  for all  $t \in \mathbb{R}$  point of continuity of  $F$  and  $\lim_{n \rightarrow +\infty} L_n(t) = L(t)$  for all  $t \in \mathbb{R}$  point of continuity of  $L$ .*

**Example 1.3.** (see [3], [1]) Let  $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$  be a left-continuous  $t$ -norm, then the operation  $\star$  defined for all  $F, L \in \Delta^+$  and for all  $t \in \mathbb{R}$  by

$$(F \star L)(t) := \sup_{s+u=t} T(F(s), L(u)) \quad (1.1)$$

is continuous at each point  $(F, L) \in \Delta^+ \times \Delta^+$  and satisfies the axioms (i)-(vi).

In all this work, we assume that  $\Delta^+$  is equipped with a continuous law  $\star$  satisfying the axioms (i)-(vi).

**Definition 1.4.** Let  $G$  be a set and let  $D : G \times G \rightarrow (\Delta^+, \star, \leq)$  be a map. We say that  $(G, D, \star)$  is a probabilistic metric space if the following axioms (i)-(iii) hold:

- (i)  $D(p, q) = H_0$  iff  $p = q$ .
- (ii)  $D(p, q) = D(q, p)$  for all  $p, q \in G$ .
- (iii)  $D(p, q) \star D(q, r) \leq D(p, r)$  for all  $p, q, r \in G$ .

## 2. PROBABILISTIC 1-LIPSCHITZ MAPS

We define probabilistic continuous functions.

**Definition 2.1.** Let  $(G, D, \star)$  be a probabilistic metric space and  $f : G \rightarrow \Delta$  be a function. We say that  $f$  is continuous at  $x \in G$  if  $\lim_n f(x_n)(t) = f(x)(t)$  for all  $t \in \mathbb{R}$  point of continuity of  $f(x)$  and all sequence  $(x_n)_n \subset G$  such that  $\lim_n D(x_n, x) = H_0$  (i.e. such that  $\lim_n D(x_n, x)(t) = H_0(t)$  for all  $t \in \mathbb{R}$ ).

Now, we introduce the notion of probabilistic 1-Lipschitz map.

**Definition 2.2.** Let  $(G, D, \star)$  be a probabilistic metric space and  $f : G \rightarrow \Delta$  be a map. We say that  $f$  is a probabilistic 1-Lipschitz map if, for all  $x, y \in G$  we have:

$$D(x, y) \star f(y) \leq f(x).$$

**Proposition 2.3.** *Every probabilistic 1-Lipschitz map is continuous.*

*Proof.* Let  $f$  be a probabilistic 1-Lipschitz map and  $(x_n)_n \subset G$  be a sequence such that  $\lim_n D(x_n, x) = H_0$ . On one hand, since  $(\Delta, \leq)$  is a complete lattice, the law  $\star$  is continuous and  $D(x_n, x) \star f(x) \leq f(x_n)$ , then  $f(x) = H_0 \star f(x) = \liminf_n D(x_n, x) \star f(x) \leq \liminf_n f(x_n)$ . Thus,  $f(x) \leq \liminf_n f(x_n)$ . On the other hand, since  $D(x_n, x) \star f(x_n) \leq f(x)$ , it follows that  $\limsup_n f(x_n) = H_0 \star \limsup_n f(x_n) = \limsup_n (D(x_n, x) \star f(x_n)) \leq f(x)$ . Thus,  $\lim_n f(x_n) = f(x)$  and so  $f$  is continuous.  $\square$

By  $Lip_\star^1(G, \Delta)$  (resp.  $Lip_\star^1(G, \Delta^+)$ ), we denotes the space of all probabilistic 1-Lipschitz maps (resp. all  $\Delta^+$ -valued 1-Lipschitz maps). For all  $x \in G$ , by  $\delta_x : G \rightarrow \Delta^+$  we denote the map  $\delta_x : y \mapsto D(y, x)$  and by  $\delta$ , we denote the operator  $\delta : x \mapsto \delta_x$ .

**Proposition 2.4.** *Let  $(G, D, \star)$  be a probabilistic metric space. Then, we have that  $\delta_a \in Lip_\star^1(G, \Delta^+)$  for each  $a \in G$  and the map  $\delta : G \rightarrow Lip_\star^1(G, \Delta^+)$  is injective.*

*Proof.* The fact that  $\delta_a \in Lip_\star^1(G, \Delta^+)$  for each  $a \in G$  follows from the property:  $D(x, y) \star D(y, a) \leq D(x, a)$  for all  $a, x, y \in G$ . Now, let  $a, b \in G$  be such that  $\delta_a = \delta_b$ . It follows that  $\delta_a(x) = \delta_b(x)$  for all  $x \in G$ . In particular, for  $x = b$  we have that  $D(a, b) = \delta_a(b) = \delta_b(b) = H_0$ , which implies that  $a = b$ .  $\square$

Let  $f \in Lip_\star^1(G, \Delta^+)$  and  $F \in \Delta^+$ , by  $\langle f, F \rangle : G \rightarrow \Delta^+$ , we denote the map defined by  $\langle f, F \rangle(x) := f(x) \star F$  for all  $x \in G$ . We easily obtain the following proposition.

**Proposition 2.5.** *Let  $(G, D, \star)$  be a probabilistic metric space. Then, for all  $f \in Lip_\star^1(G, \Delta^+)$  and all  $F \in \Delta^+$ , we have that  $\langle f, F \rangle \in Lip_\star^1(G, \Delta^+)$ .*

Recall the Lipschitz extension result of Mac Shane in [2]: if  $(X, d)$  is a metric space,  $A$  a nonempty subset of  $X$  and  $f : A \rightarrow \mathbb{R}$  is  $k$ -Lipschitz map, then there exist a  $k$ -Lipschitz map  $\tilde{f} : X \rightarrow \mathbb{R}$  such that  $\tilde{f}|_A = f$ . We give bellow an analogous of this result for probabilistic 1-Lipschitz maps.

**Theorem 2.6.** *Let  $(G, D, \star)$  be a probabilistic metric space and  $A$  be a nonempty subset of  $G$ . Let  $f : A \rightarrow \Delta$  be a probabilistic 1-Lipschitz map. Then, there exists a probabilistic 1-Lipschitz map  $\tilde{f} : G \rightarrow \Delta$  such that  $\tilde{f}|_A = f$ .*

*Proof.* We define  $\tilde{f} : G \rightarrow \Delta$  as follows: for all  $x \in G$ ,

$$\tilde{f}(x) := \sup_{a \in A} D(a, x) \star f(a).$$

We first prove that  $\tilde{f}(x) = f(x)$  for all  $x \in A$ . Indeed, let  $x \in A$ . On one hand we have  $f(x) = H_0 \star f(x) = D(x, x) \star f(x) \leq \sup_{a \in A} D(a, x) \star f(a) = \tilde{f}(x)$ . On the other hand, since  $f$  is probabilistic 1-Lipschitz on  $A$  and  $x \in A$ , then  $D(a, x) \star f(a) \leq f(x)$  for all  $a \in A$ . It follows that  $\tilde{f}(x) := \sup_{a \in A} D(a, x) \star f(a) \leq f(x)$ . Thus,  $\tilde{f}(x) = f(x)$  for all  $x \in A$ . Now, we show that  $\tilde{f}$  is probabilistic 1-Lipschitz on  $G$ . Indeed, Let  $x, y \in G$ . For all  $a \in A$  we have that  $D(a, x) \star D(x, y) \leq D(a, y)$ . So,  $D(a, x) \star f(a) \star D(x, y) \leq D(a, y) \star f(a)$ . By taking the supremum over  $a \in A$  and using axiom (vi) we get  $\tilde{f}(x) \star D(x, y) \leq \tilde{f}(y)$ . Hence,  $\tilde{f}$  is probabilistic 1-Lipschitz map on  $G$  that coincides with  $f$  on  $A$ .  $\square$

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