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INTRODUCTION TO THE PROBABILISTIC 1-LIPSCHITZ MAPS

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ABSTRACT. We introduce the notion of probabilistic 1-Lipschitz map defined on a probabilistic metric space and give an analogous to the result of Mac Shane on the extension of real-valued Lipschitz functions from a subset of a metric space to the whole space.

54E70, 47S50, 46S50

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1. Introduction

We introduce the notion of probabilistic 1-Lipschitz map defined on a probabilistic metric space and give an analogous to the result of Mac Shane (see [2]) on the extension of real-valued Lipschitz functions from a subset of a metric space to the whole space.

A distribution function is a function $F : [-\infty, +\infty] \rightarrow [0, 1]$ which is non-decreasing and left-continuous with $F(-\infty) = 0; F(+\infty) = 1$. The set of all distribution functions will be denoted by $\Delta$. The subset of $\Delta$ consisting on distributions $F$ such that $F(0) = 0$ will be denoted by $\Delta^+$. For $F, G \in \Delta^+$, the relation $F \leq G$ is meant by $F(t) \leq G(t)$, for all $t \in \mathbb{R}$. For all $a \in \mathbb{R}$, the distribution $H_a$ is defined as follow

$$H_a(t) = \begin{cases} 
0, & \text{if } t \leq a \\
1, & \text{if } t > a 
\end{cases}$$

For $a = +\infty$,

$$H_\infty(t) = \begin{cases} 
0, & \text{if } t \in [-\infty, +\infty[ \\
1, & \text{if } t = +\infty 
\end{cases}$$
It is well known that $(\Delta, \leq)$ and $(\Delta^+, \leq)$ are complete lattice with the minimal element $H_\infty$ and the maximal element $H_0$ (see [1]). Thus, for any nonempty set $I$ and any family $(F_i)_{i \in I}$ of distributions in $\Delta$ (resp. in $\Delta^+$), the function $F = \sup_{i \in I} F_i$ is also an element of $\Delta$ (resp. of $\Delta^+$).

**Axioms 1.1.** In this work, we assume that $\Delta^+$ is equipped with a law $\ast$ (a triangular function) satisfying the following axioms:

(i) $F \ast L \in \Delta^+$ for all $F, L \in \Delta^+$.
(ii) $F \ast L = L \ast F$ for all $F, L \in \Delta^+$.
(iii) $F \ast (L \ast K) = (F \ast L) \ast K$, for all $F, L, K \in \Delta^+$.
(iv) $F \ast H_0 = F$ for all $F \in \Delta^+$.
(v) $F \leq L \Longrightarrow F \ast K \leq L \ast K$ for all $F, L, K \in \Delta^+$.
(vi) Let $I$ be a set, $(F_i)_{i \in I}$ a family of distributions in $\Delta^+$ and $L \in \Delta^+$. Then, $\sup_{i \in I} (F_i \ast L) = (\sup_{i \in I} (F_i)) \ast L$.

**Definition 1.2.** We say that $\ast$ is continuous at $(F, L) \in \Delta^+ \times \Delta^+$ if $\lim_{n \to +\infty} (F_n \ast L_n)(t) = (F \ast L)(t)$ for all $t \in \mathbb{R}$ point of continuity of $F \ast L$, whenever $\lim_{n \to +\infty} F_n(t) = F(t)$ for all $t \in \mathbb{R}$ point of continuity of $F$ and $\lim_{n \to +\infty} L_n(t) = L(t)$ for all $t \in \mathbb{R}$ point of continuity of $L$.

**Example 1.3.** (see [3], [1]) Let $T : [0, 1] \times [0, 1] \to [0, 1]$ be a left-continuous $t$-norm, then the operation $\ast$ defined for all $F, L \in \Delta^+$ and for all $t \in \mathbb{R}$ by

$$ (F \ast L)(t) = \sup_{s + u = t} T(F(s), L(u)) \tag{1.1} $$

is continuous at each point $(F, L) \in \Delta^+ \times \Delta^+$ and satisfies the axioms (i)-(vi).

In all this work, we assume that $\Delta^+$ is equipped with a continuous law $\ast$ satisfying the axioms (i)-(vi).

**Definition 1.4.** Let $G$ be a set and let $D : G \times G \to (\Delta^+, \ast, \leq)$ be a map. We say that $(G, D, \ast)$ is a probabilistic metric space if the following axioms (i)-(iii) hold:

(i) $D(p, q) = H_0$ iff $p = q$.
(ii) $D(p, q) = D(q, p)$ for all $p, q \in G$.
(iii) $D(p, q) \ast D(q, r) \leq D(p, r)$ for all $p, q, r \in G$.

## 2. Probabilistic 1-Lipschitz Maps

We define probabilistic continuous functions.

**Definition 2.1.** Let $(G, D, \ast)$ be a probabilistic metric space and $f : G \to \Delta$ be a function. We say that $f$ is continuous at $x \in G$ if $\lim_{n \to \infty} f(x_n)(t) = f(x)(t)$ for all $t \in \mathbb{R}$ point of continuity of $f(x)$ and all sequence $(x_n)_{n} \in G$ such that $\lim_{n \to \infty} D(x_n, x) = H_0$ (i.e. such that $\lim_{n \to \infty} D(x_n, x)(t) = H_0(t)$ for all $t \in \mathbb{R}$).

Now, we introduce the notion of probabilistic 1-Lipschitz map.

**Definition 2.2.** Let $(G, D, \ast)$ be a probabilistic metric space and $f : G \to \Delta$ be a map. We say that $f$ is a probabilistic 1-Lipschitz map if, for all $x, y \in G$ we have:

$$ D(x, y) \ast f(y) \leq f(x). $$

**Proposition 2.3.** Every probabilistic 1-Lipschitz map is continuous.
Proof. Let $f$ be a probabilistic 1-Lipschitz map and $(x_n)_n \subset G$ be a sequence such that $\lim_n D(x_n, x) = H_0$. On one hand, since $(\Delta, \leq)$ is a complete lattice, the law $*$ is continuous and $D(x_n, x) * f(x) \leq f(x_n)$, then $f(x) = H_0 * f(x) = \lim_{n} D(x_n, x) * f(x) \leq \lim_{n} f(x_n)$. Thus, $f(x) \leq \lim_{n} f(x_n)$. On the other hand, since $D(x_n, x) * f(x_n) \leq f(x)$, it follows that $\lim_{n} f(x_n) = H_0 * \lim_{n} f(x_n) = \lim_{n} (D(x_n, x) * f(x_n)) \leq f(x)$. Thus, $\lim_{n} f(x_n) = f(x)$ and so $f$ is continuous.

By $\text{Lip}_1^1(G, \Delta)$ (resp. $\text{Lip}_1^1(G, \Delta^+)$), we denote the space of all probabilistic 1-Lipschitz maps (resp. all $\Delta^+$-valued 1-Lipschitz maps). For all $x \in G$, by $\delta_x : G \rightarrow \Delta^+$ we denote the map $\delta_x : y \mapsto D(y, x)$ and by $\delta$, we denote the operator $\delta : x \mapsto \delta_x$.

**Proposition 2.4.** Let $(G, D, *)$ be a probabilistic metric space. Then, we have that $\delta_a \in \text{Lip}_1^1(G, \Delta^+)$ for each $a \in G$ and the map $\delta : G \rightarrow \text{Lip}_1^1(G, \Delta^+)$ is injective.

**Proof.** The fact that $\delta_a \in \text{Lip}_1^1(G, \Delta^+)$ for each $a \in G$ follows from the property: $D(x, y) * D(y, a) \leq D(x, a)$ for all $a, x, y \in G$. Now, let $a, b \in G$ be such that $\delta_a = \delta_b$. It follows that $\delta_a(x) = \delta_b(x)$ for all $x \in G$. In particular, for $x = b$ we have that $D(a, b) = \delta_a(b) = \delta_b(b) = H_0$, which implies that $a = b$.

Let $f \in \text{Lip}_1^1(G, \Delta^+)$ and $F \in \Delta^+$, by $\langle f, F \rangle : G \rightarrow \Delta^+$, we denote the map defined by $\langle f, F \rangle(x) := f(x) * F$ for all $x \in G$. We easily obtain the following proposition.

**Proposition 2.5.** Let $(G, D, *)$ be a probabilistic metric space. Then, for all $f \in \text{Lip}_1^1(G, \Delta^+)$ and all $F \in \Delta^+$, we have that $\langle f, F \rangle \in \text{Lip}_1^1(G, \Delta^+)$.

Recall the Lipschitz extension result of Mac Shane in [2]: if $(X, d)$ is a metric space, $A$ a nonempty subset of $X$ and $f : A \rightarrow \mathbb{R}$ is $k$-Lipschitz map, then there exist a $k$-Lipschitz map $\tilde{f} : X \rightarrow \mathbb{R}$ such that $\tilde{f}|_A = f$. We give below an analogous of this result for probabilistic 1-Lipschitz maps.

**Theorem 2.6.** Let $(G, D, *)$ be a probabilistic metric space and $A$ be a nonempty subset of $G$. Let $f : A \rightarrow \Delta$ be a probabilistic 1-Lipschitz map. Then, there exists a probabilistic 1-Lipschitz map $\tilde{f} : G \rightarrow \Delta$ such that $\tilde{f}|_A = f$.

**Proof.** We define $\tilde{f} : G \rightarrow \Delta$ as follows: for all $x \in G$,

$$\tilde{f}(x) := \sup_{a \in A} D(a, x) * f(a).$$

We first prove that $\tilde{f}(x) = f(x)$ for all $x \in A$. Indeed, let $x \in A$. On one hand we have $f(x) = H_0 * f(x) = D(x, x) * f(x) \leq \sup_{a \in A} D(a, x) * f(a) = \tilde{f}(x)$. On the other hand, since $f$ is probabilistic 1-Lipschitz on $A$ and $x \in A$, then $D(a, x) * f(a) \leq f(x)$ for all $a \in A$. It follows that $\tilde{f}(x) := \sup_{a \in A} D(a, x) * f(a) \leq f(x)$. Thus, $\tilde{f}(x) = f(x)$ for all $x \in A$. Now, we show that $\tilde{f}$ is probabilistic 1-Lipschitz on $G$. Indeed, let $x, y \in G$. For all $a \in A$ we have that $D(a, x) * D(x, y) \leq D(a, y)$. So, $D(a, x) * f(a) * D(x, y) \leq D(a, y) * f(a)$. By taking the supremum over $a \in A$ and using axiom (vi) we get $\tilde{f}(x) * D(x, y) \leq f(y)$. Hence, $\tilde{f}$ is probabilistic 1-Lipschitz map on $G$ that coincides with $f$ on $A$. □
References

