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# On the Krein-Milman theorem for convex compact metrizable sets.

Mohammed Bachir

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*Laboratoire SAMM 4543, Université Paris 1 Panthéon-Sorbonne, Centre P.M.F. 90  
rue Tolbiac 75634 Paris cedex 13*

*Email : Mohammed.Bachir@univ-paris1.fr*

**Abstract.** The Krein-Milman theorem states that every convex compact subset of a Hausdorff locally convex topological space, is the closed convex hull of its extreme points. We prove that, in the metrizable case the situation is rather better. Indeed, we introduce a concept of "*affine exposed points*" which is intermediate between the notions of exposed points and extreme points. Then, we prove that every convex compact metrizable subset of a Hausdorff locally convex topological space, is the closed convex hull of its affine exposed points. This fails in general for not metrizable compact convex subsets.

**Keyword, phrase:** Extreme points, exposed points, Krein-Milman theorem, variational principle.

**2010 Mathematics Subject:** 46B22, 46B20, 49J50.

## 1 Introduction.

Let  $X$  be a Hausdorff locally convex topological space (in short l.c.t space, "*Hausdorff*" will be implicit),  $X^*$  denotes its topological dual. Let  $C$  be a convex subset of  $X$ , we say that a point  $x \in C$  is an extreme point of  $C$ , and write  $x \in \text{Ext}(C)$ , if and only if the following implication holds:

$$y, z \in C; x = \frac{y+z}{2} \implies x = y = z.$$

We say that a point  $x \in C$  is an exposed point of  $C$ , and write  $x \in \text{Exp}(C)$ , if there exists some continuous linear functional  $x^* \in X^*$  which attains its strict maximum over  $C$  at  $x$ . Such a functional is then said to expose  $C$  at  $x$ . Note that there can be many exposing functionals for  $x$ . It is well know that, when it exists, an exposed point is an extreme point i.e.  $\text{Exp}(C) \subseteq \text{Ext}(C)$ , but this inclusion is strict in general. If  $X$  is a dual space, a weak\* exposed point  $x$  (we write  $x \in w^*\text{Exp}(C)$ ) is to simply an exposed point by a continuous functional from the predual. We denote by  $\overline{\text{conv}}(A)$  the closed convex hull of a subset  $A$  of  $X$ .

The result in what is known as the Krein-Milman theorem (1940, [10]), asserts that if  $K$  is a convex compact subset of an l.c.t space, then  $K$  is the closed convex hull of its extreme points,

$$K = \overline{\text{conv}}(\text{Ext}(K)).$$

The Krein-Milman theorem has a partial converse known as Milman's theorem which states that if  $A$  is a subset of  $K$  and the closed convex hull of  $A$  is all of  $K$ , then every extreme point of  $K$  belongs to the closure of  $A$ ,

$$(A \subset K; K = \overline{\text{conv}}(A)) \implies \text{Ext}(K) \subset \overline{A}.$$

It is natural to ask what are the spaces in which, every convex compact subset is the closed convex hull of its exposed points. This problem was solved by Klee in [9] (1958) for normed vector space, where he proved that in normed vector spaces, every convex compact subset is the closed convex hull of its exposed points. But, Klee [9] pointed the fact that outside the normed space, some condition rather close to normability may be needed and that the metrizable is inadequate even in the separable case, mentioning the following counterexample: in the locally convex separable metrizable space  $\mathbb{R}^{\aleph_0}$ , the cube  $[-1, 1]^{\aleph_0}$  has no exposed points. We solve this problem positively in the metrizable case, by introducing a new concept of remarkable points called "*affine exposed points*" that is intermediate between the notions of exposed points and extreme points (See Definition 1 below). Our result is then the following (Theorem 3): *Every convex compact metrizable subset of a l.c.t space, is the closed convex hull of its affine exposed points.*

Let  $K$  be a convex subset of a vector space. A function  $\varphi : K \rightarrow \mathbb{R}$  is said to be affine if for all  $x, y \in K$  and  $0 \leq \lambda \leq 1$ ,  $\varphi(\lambda x + (1 - \lambda)y) = \lambda\varphi(x) + (1 - \lambda)\varphi(y)$ . The set of all continuous real-valued affine functions on a convex subset  $K$  of a topological vector space will be denoted by  $Aff(K)$ .

**Definition 1.** *Let  $K$  be a convex subset of a l.c.t space  $X$ . We say that a point  $x \in K$  is an affine exposed point of  $K$ , and write  $x \in \text{AExp}(K)$ , if there exists some affine continuous map  $\tau \in Aff(K)$  which attains its strict maximum over  $K$  at  $x$ .*

Clearly,  $\text{Exp}(K) \subseteq \text{AExp}(K) \subseteq \text{Ext}(K)$ , but these inclusions are strict in general. For example, the cube  $[-1, 1]^{\aleph_0}$  has affine exposed points but is without exposed points. A comparison of these three sets will be given in Section 2.

Let us mention here that the concept of affine exposed points seems to be very natural and useful. Indeed, we prove in Lemma 4 that if  $K$  is a convex compact metrizable subset of an l.c.t space, then a point  $Q$  of the dual unit ball  $B_{(Aff(K))^*}$  of  $Aff(K)$  is weak\* exposed if and only if there exists an affine exposed point  $k$  of  $K$  such that  $Q = \pm\delta_k$ , where  $\delta_k : \varphi \mapsto \varphi(k)$  for all  $\varphi \in Aff(K)$ .

We also introduce the following class of l.c.t spaces.

**Definition 2.** *An l.c.t space  $X$  is said to have the "Affine Exposed Points Property" (in short A.E.P.P.) if and only if every convex compact subset of  $X$  is the closed convex hull of its affine exposed points.*

Let us define

$$\Xi := \{X \text{ l.c.t space in which every compact subset is metrizable}\}.$$

The class  $\Xi$ , has been actively studied in the 80's years by several authors. This class contains of course all metrizable l.c.t spaces, in particular Fréchet spaces but is much larger. For several examples, we refer to [5] and references therein.

As a consequence, we get that every l.c.t space  $X$  from the class  $\Xi$ , has the A.E.P.P. In particular the space  $\mathbb{R}^{\aleph_0}$  has the A.E.P.P. Examples of l.c.t spaces having the A.E.P.P. who do not belong to the class  $\Xi$  are given in Remark 1. For an example of an l.c.t space without A.E.P.P. we mention the l.c.t space  $((l^1(\Gamma))^*, \text{Weak}^*)$ , where  $\Gamma$  is uncountable (See Example **D**) in Section 2). Thus, spaces having A.E.P.P. encompasses a broad class of spaces and it would be interesting to better know their properties.

The result mentioned in the abstract is based on a new version of variational principle (See Lemma 2 and Theorem 2) which also gives a new information about the set of "*ill-posed problems*" on compact metric sets (See Section 3). It is shown that the set of "*ill-posed problems*" in the compact metric framework, can be more smaller than be of the first Baire category, given by Deville, Godefroy and Zizler [6], or to be  $\sigma$ -porous, given by Deville and Rivaliski [7].

This paper is organized as follows. In Section 2 we give some examples showing that in general, exposed points, affine exposed points and extreme points are distinct notions of remarkable points. Section 3 is devoted to the proof of a new version of variational principle in the compact metric framework (Lemma 2 and Theorem 2). In Section 4, we give the proof of the result mentioned in the abstract (Theorem 3) and gives some consequences.

## 2 Examples.

Let  $K$  be a convex subset of a l.c.t space  $X$ . It is easy to see that we always have

$$\text{Exp}(K) \subseteq \text{AExp}(K) \subseteq \text{Ext}(K).$$

This section is devoted to give examples showing that these inclusions are strict in general. We also give an example of l.c.t space without the A.E.P.P.

If  $(E, \|\cdot\|)$  is a Banach space and  $E^*$  is its topological dual, the space  $X = (E^*, \text{Weak}^*)$  is a l.c.t space. It is well know that in this case we have that  $X^* = E$  (See for instance [Corollary 224., [8]]). In this case, the exposed points of a subset of  $X$  are called weak\* exposed points and the closure of a subset coincides with the weak\* closure. Recall that a Banach space  $E$  is said to be a Gâteaux differentiability space (GDS) iff each convex continuous real valued function defined on  $E$  is Gâteaux differentiable at each point of a dense subset. In [11], Phelps proved the following result.

**Theorem 1.** ([Theorem 6.2., [11] p. 95]) *A Banach space  $E$  is a GDS if and only if every weak\* compact convex subset of  $E^*$  is the weak\* closed convex hull of its weak\* exposed points.*

**Remark 1.** *Since the exposed points are in particular affine exposed points, it follows from the above theorem that the space  $(E^*, \text{Weak}^*)$  has the A.E.P.P. whenever  $E$  is a GDS. However, if  $E$  is a non separable GDS, the dual unit ball is a weak\* compact not metrizable subset. Thus, the space  $(E^*, \text{Weak}^*)$  has the A.E.P.P. but  $(E^*, \text{Weak}^*) \notin \Xi$ , whenever  $E$  is a nonseparable GDS (For example the nonseparable Hilbert spaces).*

**A) Example where  $\text{Exp}(K) \subsetneq \text{AExp}(K)$ .** Here we take the example given in the introduction. The cube  $[-1, 1]^{\mathbb{N}_0}$  in the locally convex separable metrizable space  $\mathbb{R}^{\mathbb{N}_0}$ , has no exposed points however the set of its affine exposed points is nonempty. Indeed, for example the point  $b = (1, 1, 1, \dots)$  is affine exposed in  $[-1, 1]^{\mathbb{N}_0}$  by the affine continuous map defined on  $[-1, 1]^{\mathbb{N}_0}$  by  $\varphi : (x_1, x_2, x_3, \dots) \mapsto \sum_{n \geq 0} 2^{-n} x_n$ .

A slight change of the set  $[-1, 1]^{\mathbb{N}_0}$ , gives also an example where  $\emptyset \neq \text{Exp}(K) \neq \text{AExp}(K)$ . For example we can take the convex compact set  $K := \{ta + (1-t)k/t \in [0, 1], k \in [-1, 1]^{\mathbb{N}_0}\}$ , where  $a = (-2, 0, 0, 0, \dots)$ . In this case the point  $a$  is exposed by the continuous functional  $x^* : (x_1, x_2, x_3, \dots) \mapsto -x_1$ , but the point  $b = (1, 1, 1, \dots)$  is not an exposed point. However,  $b$  is affine exposed by the affine continuous map defined on  $K$  by  $\varphi : (x_1, x_2, x_3, \dots) \mapsto \sum_{n \geq 0} 2^{-n} x_n$ .

**B) Example where  $\text{Exp}(K) = \text{AExp}(K)$ .** Let  $B_{E^*}$  be the closed unit ball of the dual of a Banach space  $E$ . It is well know from a Banach-Dieudonné result that the space  $\text{Aff}(B_{E^*})$ , (where  $B_{E^*}$  is equipped with the weak\* topology) coincides with the set  $\{\hat{x}|_{B_{E^*}} + r : x \in E, r \in \mathbb{R}\}$  where  $\hat{x}$  denotes the map  $x^* \mapsto x^*(x)$  for all  $x^* \in E^*$ . Thus, in this case the weak\* exposed points (equivalent to the concept of exposed points in the l.c.t space  $(E^*, \text{Weak}^*)$ ) and the affine exposed points of  $B_{E^*}$  coincides.

**C) Example where  $\text{AExp}(K) \subsetneq \text{Ext}(K)$ .** It suffices here to take  $K = B_{E^*}$  where  $E = l^1(\Gamma)$  ( $\Gamma$  is uncountable set) which is not a GDS. Indeed, we know that the norm  $\|\cdot\|_1$  is nowhere Gâteaux differentiable (See Example 1.4 (b) p. 3 in [11]). So from [Proposition 6.9., [11]] we get that the dual unit ball  $B_{(l^1(\Gamma))^*}$  in the l.c.t space  $((l^1(\Gamma))^*, \text{Weak}^*)$ , has no (weak\*) exposed points. It follows from Example **B**) that  $\text{AExp}(B_{(l^1(\Gamma))^*}) = \emptyset$ . However, we have that  $\text{Ext}(B_{(l^1(\Gamma))^*}) \neq \emptyset$  by the Krein-Milman theorem.

**D) Example of l.c.t space without the A.E.P.P.** The above discussion shows that the l.c.t space  $((l^1(\Gamma))^*, \text{Weak}^*)$  does not have the A.E.P.P. More generally, the l.c.t space  $(E^*, \text{Weak}^*)$  does not have the A.E.P.P. whenever  $E$  is a Banach space equipped with a nowhere Gâteaux differentiable norm.

### 3 Variational principle.

Let  $(M, d)$  be a complete metric space and  $f : M \rightarrow \mathbb{R} \cup \{+\infty\}$  be an extended real-valued function which is bounded from below and proper. By the term proper we mean that the domain of  $f$ ,  $\text{dom}(f) := \{x \in M / f(x) < +\infty\}$  is non-empty. We say that  $f$  has a strong minimum at  $x$  if  $\inf_X f = f(x)$  and  $d(x_n, x) \rightarrow 0$  whenever  $f(x_n) \rightarrow f(x)$ . The problem to find a strong minimum for  $f$ , is called *Tykhonov well-posed-problem*. Let  $(C_b(M), \|\cdot\|_\infty)$  be the space of all real-valued bounded and continuous functions on

$M$ , equipped with the sup-norm and  $(Y, \|\cdot\|_Y)$  is a Banach space included in  $C_b(M)$ . Let

$$N(f) = \{\varphi \in Y : f - \varphi \text{ does not have a strong minimum on } M\}.$$

The set  $N(f)$  is called the set of "*ill-posed problems*". The problem is to find conditions on  $Y$  under which the set  $N(f)$  is a "*small*" set. In [6], Deville, Godefroy and Zizler proved that the set  $N(f)$  is of the first Baire category in  $Y$ , and in [7], Deville and Rivalski generalize the result of Deville-Godefroy-Zizler, where they showed that the set  $N(f)$  is  $\sigma$ -porous in  $Y$ , whenever  $f$  is bounded from below, proper and lower semi continuous and  $Y$  satisfy the following conditions:

- (i)  $\|g\| \geq \|g\|_\infty$ , for all  $g \in Y$ ;
- (ii) for every natural number  $n$ , there exists a positive constant  $M_n$  such that for any point  $x \in M$  there exists a function  $h_n : X \rightarrow [0; 1]$ , such that  $h_n \in Y$ ,  $\|h_n\| \leq M_n$ ,  $h_n(x) = 1$  and  $\text{diam}(\text{supp}(h)) < \frac{1}{n}$ .

We prove in Lemma 2 that when we assume that  $(K, d)$  is compact metric space and  $(Y, \|\cdot\|_Y)$  is a separable Banach space included in  $C_b(K)$ , the set  $N(f)$  can be more smaller than  $\sigma$ -porous. In fact we prove that in this situation, the set  $N(f)$  can be covered by countably many *d.c. hypersurface* (See the definitions below). Moreover,  $Y$  does not need to satisfies the condition (ii), which is crucial in the proofs of Deville-Rivalski and Deville-Godefroy-Zizler. This will allow us to consider the space  $Y = \text{Aff}(K)$  which not satisfies the condition (ii) ( $K$  is here a convex compact subset of an l.c.t space). Our version of variational principle applied to the space of affine maps is the key for proving that the set of affine exposed points of a convex compact metrizable subset of an l.c.t space, is a nonempty set. The proof of our version of variational principle in the compact framework, is based on the use of a differentiability result of convex continuous functions on a separable Banach spaces due to Zajicek [13] and a non convex analogue to Fenchel duality introduced in [2]. This paper is also the occasion to give new applications to this duality.

We recall from [14] the following definitions.

**Definition 3.** Let  $Y, Z$  be Banach spaces,  $C \subset Y$  an open convex set, and  $F : C \rightarrow Z$  a continuous mapping. We say that  $F$  is *d.c.* (that is, *delta-convex*) if there exists a continuous convex function  $f : C \rightarrow \mathbb{R}$  such that  $y^* \circ F + f$  is convex whenever  $y^* \in Y^*$ ,  $\|y^*\| \leq 1$ .

**Definition 4.** Let  $Y$  be a Banach space and  $n \in \mathbb{N}^*$ ,  $1 \leq n < \dim Y$ . We say that  $A \subset X$  is a *d.c. surface of codimension  $n$*  if there exist an  $n$ -dimensional linear space  $F \subset X$ , its topological complement  $E$  and a *d.c.* (that is, *delta-convex*) mapping  $\varphi : E \rightarrow F$  such that  $A = \{x + \varphi(x) : x \in E\}$ . A *d.c. surface of codimension 1* will be called a *d.c. hypersurface*.

In [13], Zajicek proved that in a separable Banach space, the set  $NG(F)$  of the points where a convex continuous function  $F$  is not Gâteaux differentiable, can be covered by countably many *d.c. hypersurface*. Recall that in a separable Banach space  $Y$ , each set  $N$  which can be covered by countably many *dc hypersurface* is  $\sigma$ -lower porous, also  $\sigma$ -directionally porous; in particular it is both Aronszajn (equivalent to Gauss) null and

$\Gamma$ -null. For several properties and details about this notions of small sets we refer to [14] and references therein.

If  $(Y, \|\cdot\|_Y)$  is a Banach space included in  $C_b(M)$  with  $\|\cdot\| \geq \|\cdot\|_\infty$  and  $x \in M$ , we denote by  $\delta_x$  the evaluation map (Dirac mass) on  $Y$  at  $x$  i.e.  $\delta_x : \varphi \longrightarrow \varphi(x)$ , for all  $\varphi \in Y$ . The map  $\delta_x$  is a linear continuous functional on  $Y$  since  $\|\cdot\| \geq \|\cdot\|_\infty$ . We recall the following definition from [2].

**Definition 5.** *Let  $(M, d)$  be a complete metric space and  $(Y, \|\cdot\|_Y)$  be a Banach space included in  $C_b(M)$  with  $\|\cdot\| \geq \|\cdot\|_\infty$ . We say that the space  $Y$  has the property  $P^G$  if, for every sequence  $(x_n)_n \subset M$ , the following assertions are equivalent:*

- (i) *the sequence  $(x_n)_n$  converges in  $(M, d)$ ,*
- (ii) *the associated sequence of the Dirac masses  $(\delta_{x_n})_n$  converges in  $(Y^*, \text{Weak}^*)$ .*

The letter  $G$  in  $P^G$  is justified by the fact that the Gâteaux bornology, the Gâteaux differentiability and the weak\* topology has some connection between them. We refer to [2] for more details. The space  $C_b(M)$ , the subspace  $C_b^u(M)$  of uniformly continuous functions and the space  $Lip_b(M)$  of all bounded and Lipschitz continuous functions (equipped with their natural norms), satisfies the property  $P^G$  for any complete metric space  $(M, d)$  (see [Proposition 2.6, [2]]).

Now, what interests us in this paper is the property  $P^G$  for separable Banach spaces  $(Y, \|\cdot\|_Y)$  included in  $C_b(M)$ . As we show in the following lemma, this situation holds only when  $M$  is compact (in fact this characterizes the compact sets), but we will see in Lemma 2 that despite this restriction, we get new informations on variational principles.

**Lemma 1.** *Let  $(K, d)$  be a complete metric space and  $(Y, \|\cdot\|_Y)$  be a separable Banach space included in  $C_b(K)$ , which separate the points of  $K$  and such that  $\|\cdot\| \geq \|\cdot\|_\infty$ . Then, the following assertions are equivalent.*

- (1)  *$K$  is compact.*
- (2)  *$Y$  has the property  $P^G$ .*

*Proof.* (1)  $\implies$  (2) Suppose that  $K$  is compact and let  $(x_n)_n$  be a sequence of  $K$ . If  $(x_n)_n$  converge to some point  $x$  in  $(K, d)$ , it is clear that  $(\delta_{x_n})_n$  converge to  $\delta_x$  for the weak\* topology. Suppose now that  $(\delta_{x_n})_n$  converge to some point  $Q$  in  $Y^*$  for the weak\* topology. We prove that the sequence  $(x_n)_n$  converge in  $(K, d)$ . Indeed, suppose that  $l_1$  and  $l_2$  are two distinct cluster point of  $(x_n)_n$ . There exists two subsequences  $(y_n)_n$  and  $(z_n)_n$  such that  $(y_n)_n$  converge to  $l_1$  and  $(z_n)_n$  converge to  $l_2$ . Since  $(\delta_{x_n})_n$  converge to  $Q$  and  $(Y^*, \text{Weak}^*)$  is a Hausdorff space, it follows that  $\delta_{l_1} = Q = \delta_{l_2}$  which is a contradiction since  $Y$  separate the points of  $K$ . So, the sequence  $(x_n)_n$  has a unique cluster point, and hence it converges to some point since  $K$  is a compact metric space.

(2)  $\implies$  (1) Since  $Y$  is separable, by the Banach-Alaoglu theorem, the dual unit ball  $B_{Y^*}$  is a compact metrizable space. Let us denotes  $\delta(K) := \{\delta_k : k \in K\}$  and consider the map:

$$\begin{aligned} \delta : (K, d) &\longrightarrow (\delta(K), \text{Weak}^*) \\ x &\longmapsto \delta_x \end{aligned}$$

Since  $Y$  has the property  $P^G$ , it follows that  $(\delta(K), \text{Weak}^*)$  is a closed subspace of the compact metrizable set  $(B_{Y^*}, \text{Weak}^*)$ . Therefore,  $(\delta(K), \text{Weak}^*)$  is a Hausdorff compact

space. Since  $Y$  separate the points of  $K$ , the map  $\delta$  is one-to-one. Consequently,  $\delta : (K, d) \rightarrow (\delta(K), \text{Weak}^*)$  is a continuous and bijective map from  $(K, d)$  onto the compact space  $(\delta(K), \text{Weak}^*)$ , it is then an homeomorphism which implies that  $(K, d)$  is a compact space. □

Now, we prove the following variational principle in the compact metric framework which says that  $N(f)$  can be a very "small" subset.

**Lemma 2.** *Let  $K$  be a compact metric space and  $(Y, \|\cdot\|_Y)$  be a separable Banach space included in  $C(K)$  such that  $Y$  separate the points of  $K$  and satisfies  $\|\cdot\|_Y \geq \|\cdot\|_\infty$ . Let  $f : K \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper lower semicontinuous function. Then, the set*

$$N(f) = \{\varphi \in Y : f - \varphi \text{ does not have a strong minimum on } K\}$$

*can be covered by countably many d.c. hypersurface of  $Y$ .*

*This holds in particular when  $(Y, \|\cdot\|_Y) = (C(K), \|\cdot\|_\infty)$  or  $Y$  is any closed subspace of  $C(K)$  which separates the points of  $K$ .*

*Proof.* Consider the function  $f^\times$  defined for all  $\varphi \in Y$  by

$$f^\times(\varphi) := \sup_{x \in K} \{\varphi(x) - f(x)\}.$$

It is clear that  $f^\times$  is a convex 1-Lipschitz continuous function on  $Y$ . Since  $Y$  is separable we get from [Theorem 2; [13]] that  $f^\times$  is Gâteaux-differentiable outside a set  $N(f)$  which can be covered by countably many d.c. hypersurface. On the other hand, combining Lemma 1 and [Theorem 2.8., [2]] we get that  $f^\times$  is Gâteaux-differentiable at a point  $\varphi \in Y$  if and only if  $f - \varphi$  has a strong minimum on  $K$ . Thus, the set  $N(f)$  coincide with the set

$$\{\varphi \in Y : f - \varphi \text{ does not have a strong minimum on } K\}.$$

This gives the proof. □

**Remark 2.** *A strong and strict minimum coincides for lower semi continuous functions on a compact metric space.*

We obtain the following predual version of Stegall's variational principle.

**Corollary 1.** *Let  $E$  be a separable Banach space and  $B_{E^*}$  the closed unit ball of  $E^*$ . Let  $f : (B_{E^*}, \text{Weak}^*) \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper lower semicontinuous function. Then, the set*

$$N(f) = \{x \in E : f - \hat{x} \text{ does not have a strict minimum on } B_{E^*}\}$$

*can be covered by countably many d.c. hypersurface of  $E$ . In particular, the set of functionals from  $E$  which weak\* expose the closed unit ball  $B_{E^*}$  of  $E^*$ , has a complement in  $E$  which can be covered by countably many d.c. hypersurface.*

*Proof.* Since  $E$  is a separable Banach space, the closed unit ball  $B_{E^*}$  of  $E^*$  is a weak\* compact metrizable set, by the Banach-Alaoglu theorem. We conclude by applying Lemma 2 with the compact metrizable set  $K = (B_{E^*}, \text{Weak}^*)$  and the Banach space  $Y = (\mathcal{A}(B_{E^*}), \|\cdot\|_\infty)$  which is isometrically isomorphic to  $(E, \|\cdot\|)$ , where  $\mathcal{A}(B_{E^*}) := \{\hat{x}|_{B_{E^*}}/x \in E\}$  and  $\hat{x} : x^* \mapsto x^*(x)$  for all  $x^* \in E^*$ . The second part is given by taking  $f = 0$ .  $\square$

Now, we give the main result of this section.

**Theorem 2.** *Let  $K$  be a compact metrizable convex subset of a l.c.t space  $X$  and  $f : K \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper lower semi-continuous function. Then the set*

$$N(f) := \{\varphi \in \text{Aff}(K) : f - \varphi \text{ does not have a strong minimum on } K\}$$

*can be covered by countably many d.c hypersurface of  $(\text{Aff}(K), \|\cdot\|_\infty)$ .*

*Proof.* We use Lemma 2 with  $Y = \text{Aff}(K)$ . Since  $(\text{Aff}(K), \|\cdot\|_\infty)$  is a closed Banach subspace of the separable Banach space  $(C(K), \|\cdot\|_\infty)$ , it is separable. On the other hand, by the Hahn-Banach theorem,  $\text{Aff}(K)$  separate the points of  $K$ , since it contains the set  $\{x^*|_K : x^* \in X^*\}$ . So, from Lemma 2, the set

$$N(f) = \{\varphi \in \text{Aff}(K) : f - \varphi \text{ does not have a strong minimum on } K\}$$

can be covered by countably many d.c hypersurface of  $(\text{Aff}(K), \|\cdot\|_\infty)$ .  $\square$

Let  $K$  be a convex compact subset of an l.c.t space. Clearly, all translates of continuous linear functionals are elements of  $\text{Aff}(K)$ , but the converse is not true in general (see [12] page 22.). However, we do have the following relationship.

**Proposition 1.** *([12], Proposition 4.5) Assume that  $K$  is a compact convex subset of an l.c.t space  $X$ , then*

$$L(K) := \left\{ a \in \text{Aff}(K) : a = r + x^*|_K \text{ for some } x^* \in X^* \text{ and some } r \in \mathbb{R} \right\}$$

*is dense in  $(\text{Aff}(K), \|\cdot\|_\infty)$ , where  $\|\cdot\|_\infty$  denotes the norm of uniform convergence.*

As it is given in Example **B**) Section 2, there exists situations where the sets  $L(K)$  and  $\text{Aff}(K)$  coincides, for instance if  $K = B_{E^*}$  is the dual unit ball of a Banach space. There exist also situations, where  $L(K)$  is a very "small" subset of  $\text{Aff}(K)$ . Indeed, if  $K$  is a compact metrizable subset of a l.c.t space  $X$ , without exposed points (for example if  $K = [-1, 1]^{\aleph_0}$  in  $\mathbb{R}^{\aleph_0}$ ), then from Theorem 2, we get that  $L(K)$  can be covered by countably many d.c hypersurface of  $(\text{Aff}(K), \|\cdot\|_\infty)$ .

## 4 Affine exposed points and consequences.

Now, we give the main result of this section.

**Theorem 3.** *Let  $K$  be a convex compact metrizable subset of a l.c.t space  $X$ . Then,  $\text{AExp}(K) \neq \emptyset$  and  $K$  is the closed convex hull of its affine exposed points*

$$K = \overline{\text{conv}}(\text{AExp}(K)).$$

*Moreover, the set of all affine maps which affine expose  $K$  at some point, has a complement which can be covered by countably many d.c hypersurface in  $(\text{Aff}(K), \|\cdot\|_\infty)$ .*

*Proof.* From Theorem 2 (applied with  $f = 0$ ), we get that the set of affine maps which affine expose  $K$  at some point, has a complement which can be covered by countably many d.c hypersurface in  $(\text{Aff}(K), \|\cdot\|_\infty)$ , in particular  $\text{AExp}(K) \neq \emptyset$ . Since  $K$  is convex and closed set, it is clear that  $\overline{\text{conv}}(\text{AExp}(K)) \subset K$ . Now, let us prove that  $K = \overline{\text{conv}}(\text{AExp}(K))$ . Suppose towards a contradiction that there exists  $k_0 \in K \setminus \overline{\text{conv}}(\text{AExp}(K))$ . By the Hahn-Banach theorem, there exists  $x_0^* \in X^* \setminus \{0\}$  and  $r \in \mathbb{R}$  such that

$$\sup\{x_0^*(k) : k \in \overline{\text{conv}}(\text{AExp}(K))\} < r < x_0^*(k_0).$$

From Theorem 2 (applied with  $f = x_0^*|_K \in \text{Aff}(K)$ ), we can find  $\varphi \in \text{Aff}(K)$  which affine exposes  $K$  at some point  $k_1 \in \text{AExp}(K)$  and such that  $\varphi$  is close (with respect to the sup-norm) to  $x_0^*|_K$  in  $(\text{Aff}(K), \|\cdot\|_\infty)$ . Hence,  $\varphi$  satisfies

$$\sup\{\varphi(k) : k \in \overline{\text{conv}}(\text{AExp}(K))\} < r < \varphi(k_0). \quad (1)$$

On the other hand

$$\varphi(k_0) \leq \sup\{\varphi(k) : k \in K\} = \varphi(k_1)$$

which is a contradiction with (1), since  $k_1 \in \text{AExp}(K)$ .  $\square$

Note that the above theorem fails for convex compact sets which are not metrizable. A counterexample is the closed unit ball  $B_{l^\infty(\Gamma)}$  in the l.c.t  $(l^\infty(\Gamma), \text{Weak}^*)$ , where  $\Gamma$  is an uncountable set. Indeed, in this case we have that  $\text{AExp}(B_{l^\infty(\Gamma)}) = \emptyset$  (See Example D) in Section 2).

Let us denote by  $B_{(\text{Aff}(K))^*}$  the dual unit ball of the Banach space  $(\text{Aff}(K), \|\cdot\|_\infty)$ . We give bellow the connection between the weak\* exposed points of  $B_{(\text{Aff}(K))^*}$  and the affine exposed points of  $K$ . We need the following lemma from [4].

**Lemma 3.** *Let  $Z$  be a Banach space and  $h, k : Z \rightarrow \mathbb{R}$  be two continuous and convex functions. Suppose that the function  $z \rightarrow l(z) := \max(h(z), k(z))$  is Fréchet (respectively, Gâteaux) differentiable at some point  $z_0 \in Z$ . Then either  $h$  or  $k$  (maybe both  $h$  and  $k$ ) is Fréchet (respectively, Gâteaux) differentiable at  $z_0$  and  $l'(z_0) = h'(z_0)$  or  $l'(z_0) = k'(z_0)$ .*

*Proof.* We give the proof for the Fréchet differentiability, the Gâteaux differentiability is similar. Suppose without loss of generality that  $l(z_0) = h(z_0)$  and let us prove that  $h$  is Fréchet differentiable at  $z_0$  and that  $l'(z_0) = h'(z_0)$ . For each  $z \neq 0$  we have:

$$0 \leq \frac{h(z_0 + z) + h(z_0 - z) - 2h(z_0)}{\|z\|} \leq \frac{l(z_0 + z) + l(z_0 - z) - 2l(z_0)}{\|z\|}.$$

Since  $l$  is convex and Fréchet differentiable at  $z_0$ , then the right-hand side in the above inequalities, tends to 0 when  $z$  tends to 0. This implies that  $h$  is Fréchet differentiable at  $z_0$  by the convexity of  $h$ . Now, if we denote  $f = h - l$ , then  $f(z_0) = 0$ ,  $f \leq 0$  and  $f'(z_0)$  exists. Thus, for all  $z \in Z$

$$f'(z_0)(z) = \lim_{t \rightarrow 0^+} \frac{1}{t} (f(z_0 + tz) - f(z_0)) \leq 0.$$

This implies that  $f'(z_0) = 0$ . Thus  $h'(z_0) = l'(z_0)$ .  $\square$

Now, we prove in the following lemma which shows that the concept of affine exposed points is the natural concept to describe the weak\* exposed point of the dual unit ball  $B_{(Aff(K))^*}$ .

**Lemma 4.** *Let  $K$  be a compact metrizable convex subset of a l.c.t space  $X$ . Then, the following assertions are equivalent.*

- (1) *A point  $Q \in B_{(Aff(K))^*}$  is a weak\* exposed point*
- (2) *there exists an affine exposed point  $k \in \text{AExp}(K)$  such that  $Q = \pm \delta_k$ .*

*Proof.* (1)  $\implies$  (2). Let  $Q \in w^*\text{Exp}(B_{(Aff(K))^*})$ , so there exists  $\varphi \in Aff(K)$  which expose  $B_{(Aff(K))^*}$  at  $Q$ . It follows from [Proposition 6.9., [11]] that the norm  $\|\cdot\|_\infty$  is Gâteaux differentiable at  $\varphi$  with Gâteaux derivative equal to  $Q$ . On the other hand it is clear that  $\|\psi\|_\infty = \max(0^\times(\psi), 0^\times(-\psi))$  for all  $\psi \in Aff(K)$ , where  $0^\times(\psi) = \sup_{k \in K} \varphi(k)$  for all  $\varphi \in Aff(K)$ . Thus, from Lemma 3 we have that either  $\psi \mapsto 0^\times(\psi)$  or  $\psi \mapsto 0^\times(-\psi)$  is Gâteaux differentiable at  $\varphi$  with Gâteaux derivative equal to  $Q$ . Suppose in the first case that is the function  $\psi \mapsto 0^\times(\psi)$  which is Gâteaux differentiable at  $\varphi$  with Gâteaux derivative equal to  $Q$ . Thus, from Lemma 1 and [Theorem 2.8 [2]] applied with the space  $Y = Aff(K)$  and the function  $f = 0$ , we get that there exists  $k \in K$  such that  $\varphi$  has a strong maximum at  $k$  and  $Q = \delta_k$ . Thus, in this case  $k$  is affine exposed by  $\varphi$  and  $Q = \delta_k$ . For the second case, where it is the function  $\psi \mapsto 0^\times(-\psi)$  which is Gâteaux differentiable at  $\varphi$  with Gâteaux derivative equal to  $Q$ , in a similar way, using Lemma 1, [Theorem 2.8 [2]] and the chain rule formula we obtain that there exists some  $k \in K$  such that  $-\varphi$  has a strong maximum at  $k$  (so that  $k$  is affine exposed point) and  $Q = -\delta_k$ .

(2)  $\implies$  (1). Suppose that  $k \in \text{AExp}(K)$ . There exists  $\varphi \in Aff(K)$  which affine exposes  $k$ . Thus  $-\varphi$  has a strict minimum at  $k$ , equivalent to a strong minimum at  $k$ , since  $K$  is compact metrizable. We can find a real number  $r$  such that  $-(\varphi + r)$  has also a strong minimum at  $k$  and such that  $\varphi + r > 1$  on  $K$ . Hence, the function  $0^\times$  coincides with  $\|\cdot\|_\infty$  on an open neighborhood of  $\varphi + r \in Aff(K)$ . Since  $-(\varphi + r)$  has a strong minimum at  $k$ , [Theorem 2.8 [2]] asserts that  $0^\times$  and so also  $\|\cdot\|_\infty$  is Gâteaux differentiable at  $\varphi + r$  with Gâteaux derivative equal to  $\delta_k$ . It follows from [Proposition 6.9., [11]], that  $\delta_k$  is weak\* exposed by  $\varphi + r$ . Thus  $\delta_k \in w^*\text{Exp}(B_{(Aff(K))^*})$ . By the symmetry of  $B_{(Aff(K))^*}$ , we also have that  $-\delta_k \in w^*\text{Exp}(B_{(Aff(K))^*})$ .  $\square$

**Corollary 2.** *Let  $K$  be a compact metrizable convex subset of a l.c.t space  $X$ . Then,*

$$w^*\text{Exp}(B_{(Aff(K))^*}) = \pm \delta(\text{AExp}(K)),$$

and

$$B_{(Aff(K))^*} = \overline{\text{conv}}^{w^*}(\pm \delta(\text{AExp}(K))),$$

where  $\pm\delta(\text{AExp}(K)) := \{\pm\delta_k/k \in \text{AExp}(K)\}$ .

*Proof.* The first part is given by Lemma 4. Now, since  $(\text{Aff}(K), \|\cdot\|_\infty)$  is separable, the weak\* compact set  $(B_{(\text{Aff}(K))^*}, \text{Weak}^*)$  is metrizable. Thus, from Theorem 3 applied to the convex compact metrizable set  $(B_{(\text{Aff}(K))^*}, \text{Weak}^*)$ , shows that

$$B_{(\text{Aff}(K))^*} = \overline{\text{conv}}^{w^*}(\text{AExp}(B_{(\text{Aff}(K))^*})).$$

Since  $\text{AExp}(B_{(\text{Aff}(K))^*})$  and  $w^*\text{Exp}(B_{(\text{Aff}(K))^*})$  coincides (See Example D) in Section 2), we obtain that

$$B_{(\text{Aff}(K))^*} = \overline{\text{conv}}^{w^*}(w^*\text{Exp}(B_{(\text{Aff}(K))^*})) = \overline{\text{conv}}^{w^*}(\pm\delta(\text{AExp}(K))).$$

This concludes the proof. □

We obtain the following corollaries.

**Corollary 3.** *Let  $E$  be a Banach space.*

(1) *Let  $K$  be a convex weak\* compact metrizable subset of  $E^*$ , then*

$$K = \overline{\text{conv}}^{\text{Weak}^*}(\text{AExp}(K)).$$

(2) *Let  $K$  be a convex weak compact metrizable subset of  $E$ , then*

$$K = \overline{\text{conv}}^{\|\cdot\|}(\text{AExp}(K)).$$

*Proof.* Since  $(E^*, \text{Weak}^*)$  and  $(E, \text{Weak})$  are l.c.t spaces, the part (1) is a direct consequence of Theorem 3. Also from Theorem 3 we get that  $K = \overline{\text{conv}}^{\text{Weak}^*}(\text{AExp}(K))$ . The part (2) follows then from Mazur's lemma on the coincidence of weak and norm closure for convex sets. □

Recall that the class  $\Xi$  consists on all l.c.t space in which every compact subset is metrizable. We obtain immediately from Theorem 3 the following corollary.

**Corollary 4.** *Every space from the class  $\Xi$ , has the A.E.P.P.*

**Examples 1.** *Immediate examples.*

(1) *Every Fréchet space has the A.E.P.P.*

(2) *Every convex closed and bounded subset of a Fréchet-Montel space is the closed convex hull of its affine exposed points (in Fréchet-Montel space, any closed bounded set is compact metrizable).*

A classical example of a Fréchet-Montel space is the space  $C^\infty(\Omega)$  of smooth functions on an open set  $\Omega$  in  $\mathbb{R}^n$ . For examples of not metrizable spaces which belongs to the class  $\Xi$ , we have for example,

**Proposition 2.** *Let  $E$  be a separable Banach space. Then  $(E^*, \text{Weak}^*)$  and  $(E, \text{Weak})$  belongs to the class  $\Xi$ , but are not metrizable.*

*Proof.* It is well known that the whole spaces  $(E^*, \text{Weak}^*)$  and  $(E, \text{Weak})$  are not metrizable. It is also well known that a Banach space  $E$  is separable iff every compact subset of  $(E^*, \text{Weak}^*)$  is metrizable. Thus,  $(E^*, \text{Weak}^*) \in \Xi$ . For the space  $(E, \text{Weak})$ , let  $K$  be a weak compact subset of  $E$ . Since  $E$  is separable, then  $K$  is also separable. Now, consider  $K$  as a subset of  $E^{**}$  by the canonical embedding, we get that  $K$  is norm separable and weak\* compact subset of  $E^{**}$ , which implies from [Lemma 2, [3]] that  $K$  is weak\* metrizable in  $E^{**}$ . In other words,  $K$  is weak metrizable. Thus  $(E, \text{Weak}) \in \Xi$ .  $\square$

Several others not trivial examples of spaces from  $\Xi$  can be found in [5].

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