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On the Krein-Milman theorem for convex compact metrizable sets.

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Abstract. The Krein-Milman theorem states that a convex compact subset of a Hausdorff locally convex topological space, is the closed convex hull of its extreme points. We prove that, in the metrizable case the situation is rather better. Indeed, every convex compact metrizable subset of a Hausdorff locally convex topological space, is the closed convex hull of its exposed points. This fails in general for not metrizable compact convex subsets.

Keyword, phrase: Extreme points, exposed points, Krein-Milman theorem, variational principle.

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1 Introduction.

Let X be a Hausdorff locally convex topological space (in short l.c.t space, "*Hausdorff*" will be implicit), X^* denotes its topological dual. Let C be a convex subset of X , we say that a point $x \in C$ is an extreme point of C , and write $x \in \text{Ext}(C)$, if and only if the following implication holds:

$$y, z \in C; x = \frac{y+z}{2} \implies x = y = z.$$

We say that a point $x \in C$ is an exposed point of C , and write $x \in \text{Exp}(C)$, if there exists some continuous linear functional $x^* \in X^*$ which attains its strict maximum over C at x . Such a functional is then said to expose C at x . Note that there can be many exposing functionals for x . It is well know that, when it exists, an exposed point is an extreme point i.e. $\text{Exp}(C) \subseteq \text{Ext}(C)$, but this inclusion is strict in general. If X is a dual space, a weak* exposed point is to simply an exposed point by a continuous functional from the predual. We denote by $\overline{\text{conv}}(A)$ the closed convex hull of a subset A of X .

The result in what is known as the Krein-Milman theorem (1940, [8]), asserts that if K is a convex compact subset of an l.c.t space, then K is the closed convex hull of

its extreme points,

$$K = \overline{\text{conv}}(\text{Ext}(K)).$$

The Krein-Milman theorem has a partial converse known as Milman's theorem which states that if A is a subset of K and the closed convex hull of A is all of K , then every extreme point of K belongs to the closure of A ,

$$(A \subset K; K = \overline{\text{conv}}(A)) \implies \text{Ext}(K) \subset \overline{A}.$$

It is natural to ask what are the spaces in which, every convex compact subset is the closed convex hull of its exposed points instead of its extreme points. There are known examples where this is true, for instance the spaces with the Radon-Nikodym property ([Theorem 5.21., [9]]. Our question was mainly motivated by a result of Phelps [Theorem 6.2., [9]], where it is shown that a Banach space E is a Gâteaux differentiability space (i.e. every convex continuous function on E is Gâteaux differentiable on a dense subset) iff every convex weak* compact subset of E^* is the weak* closed convex hull of its weak* exposed points. This leads us to introduce the following definition.

Definition 1. *An l.c.t space X is said to have the "Exposed Points Property" (in short E.P.P.) if and only if every convex compact subset of X is the closed convex hull of its exposed points.*

In the line of this definition, the l.c.t space (E^*, Weak^*) (where E is a Gâteaux differentiability space), has the E.P.P. Now, let us define

$$\Xi := \{X \text{ l.c.t space in which every compact subset is metrizable}\}.$$

The class Ξ , has been actively studied in the 80's years by several authors. This class contains of course all metrizable l.c.t spaces, in particular Fréchet spaces but is much larger. For several examples, we refer to [4] and references therein.

The main result of this paper (Theorem 3) proves that:

Every convex compact metrizable subset of an l.c.t space is the closed convex hull of its exposed points.

As a consequence, we get that every l.c.t space X from the class Ξ , has the E.P.P. Note that the class Ξ and the class

$$\Upsilon := \{(E^*, \text{Weak}^*)/E \text{ is a Gâteaux differentiability space}\}$$

are two distinct class of spaces with the E.P.P. In fact, we can see that when E is a separable Banach space, then (E^*, Weak^*) and (E, Weak) belongs to the class Ξ (Proposition 3), but if H is a non separable Hilbert space, then $(H^*, \text{Weak}^*) \cong (H, \text{Weak})$ belongs to Υ but not to Ξ . For an example of an l.c.t space without E.P.P. we mention the space $((l^\infty)^*, \text{Weak}^*)$ (See Section 3). Thus, spaces having E.P.P. encompasses a broad class of spaces and it would be interesting to better know their properties.

Our main result is based on a new version of variational principle (Theorem 1) from which we also get the following differentiability result (Corollary 1) revealing a difference between a Gâteaux differentiability space and a general Banach space:

In any Banach space E and for every convex continuous function f on E , if $\overline{\text{dom}(f^)}^{\text{Weak}^*}$ is weak* compact metrizable subset of E^* , then f has points of Gâteaux differentiability, where $\text{dom}(f^*)$ denotes the domain of the Fenchel conjugate f^* of f .*

Finally, the results of this paper also gives a new information about the set of "*ill-posed problems*" on compact metric sets (See Section 2). It is shown that the set of "*ill-posed problems*" in the compact metric framework, can be more smaller than be of the first Baire category, given by Deville Godefroy and Zizler [5], or to be σ -porous, given by Deville and Rivalski [6].

This paper is organized as follows. Section 2 is devoted to the proof of a variational principle (Theorem 1) and its consequence Corollary 1. In Section 3 we give the proof of Theorem 3 and some consequences, using reults of Section 2.

2 Variational principle.

Let (M, d) be a complete metric space and $f : M \rightarrow \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function which is bounded from below and proper. By the term proper we mean that the domain of f , $dom(f) := \{x \in M / f(x) < +\infty\}$ is non-empty. We say that f has a strong minimum at x if $\inf_X f = f(x)$ and $d(x_n, x) \rightarrow 0$ whenever $f(x_n) \rightarrow f(x)$. The problem to find a strong minimum for f , is called *Tykhonov well-posed-problems*.

Now, let $(C_b(M), \|\cdot\|_\infty)$ be the space of all real-valued bounded and continuous functions on M , equipped with the sup-norm and $(Y, \|\cdot\|_Y)$ is a Banach space included in $C_b(M)$. Let

$$N(f) = \{\varphi \in Y : f - \varphi \text{ does not have a strong minimum on } M\}.$$

The set $N(f)$ is called the set of "*ill-posed problems*". The problem is to find conditions on Y under which the set $N(f)$ is a "*small*" set. In [5], Deville, Godefroy and Zizler proved that the set $N(f)$ is of the first Baire category in Y , whenever f is bounded from below, proper and lower semi continuous and Y satisfy the following conditions:

- (i) $\|g\| \geq \|g\|_\infty$, for all $g \in Y$;
- (ii) for every natural number n , there exists a positive constant M_n such that for any point $x \in M$ there exists a function $h_n : X \rightarrow [0; 1]$, such that $h_n \in Y$, $\|h_n\| \leq M_n$, $h_n(x) = 1$ and $diam(supp(h)) < \frac{1}{n}$.

In [6], Deville and Rivalski generalize the result of Deville-Godefroy-Zizler, where they showed that the set $N(f)$ is σ -porous in Y . We prove in Lemma 2 that when we assume that (M, d) is compact metric space, the set $N(f)$ can be more smaller than σ -porous. In fact we prove that $N(f)$ can be covered by countably many *d.c. hypersurface* (See the definitions below). Moreover, Y does not need to satisfies the condition (ii), which is crucial in the proofs of Deville-Rivalski and Deville-Godefroy-Zizler. This will allow us to consider the space $Y = Aff(K)$ (which not satisfies the condition (ii)) of all real valued continuous affine maps defined on a convex compact metrizable subset of an l.c.t space. Our version of variational principle applied to the space $Y = Aff(K)$ is the key for proving the main result (Theorem 3). Our approach is based on the use of a differentiability result of convex continuous functions on a separable Banach spaces due to Zajicek [11] and a non convex analogue to Fenchel duality introduced in [2]. This paper is also the occasion to give new applications to this duality.

We recall from [12] the following definitions.

Definition 2. Let Y, Z be Banach spaces, $C \subset Y$ an open convex set, and $F : C \rightarrow Z$ a continuous mapping. We say that F is d.c. (that is, delta-convex) if there exists a continuous convex function $f : C \rightarrow \mathbb{R}$ such that $y^* \circ F + f$ is convex whenever $y^* \in Y^*$, $\|y^*\| \leq 1$.

Definition 3. Let Y be a Banach space and $n \in \mathbb{N}^*$, $1 \leq n < \dim Y$. We say that $A \subset X$ is a d.c. surface of codimension n if there exist an n -dimensional linear space $F \subset X$, its topological complement E and a d.c. (that is, delta-convex) mapping $\varphi : E \rightarrow F$ such that $A = \{x + \varphi(x) : x \in E\}$. A d.c. surface of codimension 1 will be called a d.c. hypersurface.

In [11], Zajicek proved that in a separable Banach space, the set $NG(F)$ of the points where a convex continuous function F is not Gâteaux differentiable, can be covered by countably many d.c. hypersurface. Recall that in a separable Banach space Y , each set N which can be covered by countably many dc hypersurface is σ -lower porous, also σ -directionally porous; in particular it is both Aronszajn (equivalent to Gauss) null, Γ -null and nowhere dense. For several properties and details about this notions of small sets we refer to [12] and references therein.

If $(Y, \|\cdot\|_Y)$ is a Banach space included in $C_b(M)$ with $\|\cdot\| \geq \|\cdot\|_\infty$ and $x \in M$, we denote by δ_x the evaluation map (Dirac mass) on Y at x i.e. $\delta_x : \varphi \rightarrow \varphi(x)$, for all $\varphi \in Y$. The map δ_x is a linear continuous functional on Y since $\|\cdot\| \geq \|\cdot\|_\infty$. We recall the following definition from [2].

Definition 4. Let (M, d) be a complete metric space and $(Y, \|\cdot\|_Y)$ be a Banach space included in $C_b(M)$ with $\|\cdot\| \geq \|\cdot\|_\infty$. We say that the space Y has the property P^G if, for every sequence $(x_n)_n \subset M$, the following assertions are equivalent:

- (i) the sequence $(x_n)_n$ converges in (M, d) ,
- (ii) the associated sequence of the Dirac masses $(\delta_{x_n})_n$ converges in (Y^*, Weak^*) .

The letter G in P^G is justified by the fact that the Gâteaux bornology, the Gâteaux differentiability and the weak* topology has some connection between them. We refer to [2] for more details. The space $C_b(M)$, the subspace $C_b^u(M)$ of uniformly continuous functions and the space $Lip_b(M)$ of all bounded and Lipschitz continuous functions (equipped with their natural norms), satisfies the property P^G for any complete metric space (M, d) (see [Proposition 2.6, [2]]).

Now, what interests us in this paper is the property P^G for separable Banach spaces $(Y, \|\cdot\|_Y)$ included in $C_b(M)$. As we show in the following lemma, this situation holds only when M is compact (in fact this characterizes the compact sets), but we will see later that despite this restriction, we get new informations on "variational principles".

Lemma 1. Let (K, d) be a complete metric space and $(Y, \|\cdot\|_Y)$ be a separable Banach space included in $C_b(K)$, which separate the points of X and such that $\|\cdot\| \geq \|\cdot\|_\infty$. Then, the following assertions are equivalent.

- (1) K is compact.
- (2) Y has the property P^G .

Proof. (1) \implies (2) Suppose that K is compact and let $(x_n)_n$ be a sequence of K . If $(x_n)_n$ converge to some point x in (K, d) , it is clear that $(\delta_{x_n})_n$ converge to δ_x for the

weak* topology. Suppose now that $(\delta_{x_n})_n$ converge to some point Q in Y^* for the weak* topology. We prove that the sequence $(x_n)_n$ converge in (K, d) . Indeed, suppose that l_1 and l_2 are two distinct cluster point of $(x_n)_n$. There exists two subsequences $(y_n)_n$ and $(z_n)_n$ such that $(y_n)_n$ converge to l_1 and $(z_n)_n$ converge to l_2 . Since $(\delta_{x_n})_n$ converge to Q and (Y^*, Weak^*) is a Hausdorff space, it follows that $\delta_{l_1} = Q = \delta_{l_2}$ which is a contradiction since Y separate the point of K . So, the sequence $(x_n)_n$ has a unique cluster point, and hence it converges to some point since K is a compact metric space.

(2) \implies (1) Since Y is separable, by the Banach-Alaoglu theorem, the dual unit ball B_{Y^*} is a compact metrizable space. Consider the map:

$$\begin{aligned} \delta : (K, d) &\rightarrow (\delta(K), \text{Weak}^*) \\ x &\mapsto \delta_x \end{aligned}$$

where, $\delta(K) := \{\delta_k : k \in K\}$. Since Y has the property P^G , it follows that $(\delta(K), \text{Weak}^*)$ is a closed subspace of the compact metrizable set (B_{Y^*}, Weak^*) . Therefore, $(\delta(K), \text{Weak}^*)$ is a Hausdorff compact space. Since Y separate the point of K , the map δ is one-to-one. Consequently, $\delta : (K, d) \rightarrow (\delta(K), \text{Weak}^*)$ is a continuous and bijective map from (K, d) onto the compact space $(\delta(K), \text{Weak}^*)$, it is then an homeomorphism which implies that (K, d) is a compact space. □

Now, we prove the following variational principle in the compact metric framework which says that $N(f)$ can be a very "small" subset.

Lemma 2. *Let K be a compact metric space and $(Y, \|\cdot\|_Y)$ be a separable Banach space included in $C(K)$ such that Y separate the points of K and satisfies $\|\cdot\|_Y \geq \|\cdot\|_\infty$. Let $f : K \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Then, the set*

$$N(f) = \{\varphi \in Y : f - \varphi \text{ does not have a strong minimum on } K\}$$

can be covered by countably many d.c. hypersurface of Y .

This holds in particular when $(Y, \|\cdot\|_Y) = (C(K), \|\cdot\|_\infty)$ or Y is any closed subspace of $C(K)$ which separate the points of K .

Proof. Consider the function f^\times defined for all $\varphi \in Y$ by

$$f^\times(\varphi) := \sup_{x \in K} \{\varphi(x) - f(x)\}.$$

It is clear that f^\times is a convex 1-Lipschitz continuous function on Y . Since Y is separable we get from [Theorem 2; [11]] that f^\times is Gâteaux-differentiable outside a set $N(f)$ which can be covered by countably many d.c hypersurface. On the other hand, combining Lemma 1 and [Theorem 2.8., [2]] we get that f^\times is Gâteaux-differentiable at a point $\varphi \in Y$ if and only if $f - \varphi$ has a strong minimum on K . Thus, the set $N(f)$ coincide with the set

$$\{\varphi \in Y : f - \varphi \text{ does not have a strong minimum on } K\}.$$

This gives the proof. □

Remark 1. *A strong and strict minimum coincides for lower semi continuous functions on a compact metric space.*

Let K and C be convex subsets of vector spaces. A function $T : K \rightarrow C$ is said to be affine if for all $x, y \in K$ and $0 \leq \lambda \leq 1$, $T(\lambda x + (1 - \lambda)y) = \lambda T(x) + (1 - \lambda)T(y)$. The set of all continuous real-valued affine functions on a convex subset K of a topological vector space will be denoted by $Aff(K)$. Clearly, all translates of continuous linear functionals are elements of $Aff(K)$, but the converse is not true in general (see [10] page 22.). However, we do have the following relationship.

Proposition 1. *([10], Proposition 4.5) Assume that K is a compact convex subset of a Hausdorff locally convex topological space X then*

$$L(K) := \left\{ a \in Aff(K) : a = r + x_{|K}^* \text{ for some } x^* \in X^* \text{ and some } r \in \mathbb{R} \right\}$$

is dense in $(Aff(K), \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ denotes the norm of uniform convergence.

Now, we give the variational principle for the space of affine maps.

Theorem 1. *Let K be a compact metrizable convex subset of a l.c.t space X and $f : K \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous function. Then*

$$N(f) = \{\varphi \in Aff(K) : f - \varphi \text{ does not have a strict minimum on } K\}$$

can be covered by countably many d.c hypersurface of $(Aff(K), \|\cdot\|_\infty)$. Consequently, for all $\varepsilon > 0$, there exists $x^ \in X^*$ such that*

- (1) $\|x_{|K}^*\|_\infty := \sup_{k \in K} |x^*(k)| < \varepsilon$ and
- (2) $f - x_{|K}^*$ has a strict minimum on K .

Proof. We will initially use Lemma 2 with $Y = Aff(K)$. Since $(Aff(K), \|\cdot\|_\infty)$ is a closed Banach subspace of the separable Banach space $(C(K), \|\cdot\|_\infty)$, it is separable. On the other hand, by the Hahn-Banach theorem, $Aff(K)$ separate the points of K , since it contains the set $\{x_{|K}^* : x^* \in X^*\}$. So, from Lemma 2, the set

$$N(f) = \{\varphi \in Aff(K) : f - \varphi \text{ does not have a unique minimum on } K\}$$

can be covered by countably many d.c hypersurface of $(Aff(K), \|\cdot\|_\infty)$.

Now, replacing K by $K - k_0$ and f by $\tilde{f}(\cdot) := f(\cdot + k_0)$ for some $k_0 \in K$, we can assume without loss of generality that $0 \in K$. let $\varepsilon > 0$, and let us denote by $B_A(0, \frac{\varepsilon}{2})$ the closed ball of $(Aff(K), \|\cdot\|_\infty)$ centred at 0 with radius $\frac{\varepsilon}{2}$. By Proposition 1 we have that $\overline{L(K)} \cap B_A(0, \frac{\varepsilon}{2}) = B_A(0, \frac{\varepsilon}{2})$, since $L(K)$ is dense in $(Aff(K), \|\cdot\|_\infty)$. So, we have that $L(K) \cap B_A(0, \frac{\varepsilon}{2}) \not\subseteq N(f)$, since $N(f)$ is in particular nowhere dense. Thus, there exists some $x^* \in X^*$ and $r \in \mathbb{R}$ such that $x_{|K}^* + r \in B_A(0, \frac{\varepsilon}{2})$ but $x_{|K}^* + r \notin N(f)$. In other words, $\|x_{|K}^* + r\|_\infty < \frac{\varepsilon}{2}$ and $f - (x_{|K}^* + r)$ has a unique minimum on K . Since $0 \in K$, we get that $|r| \leq \|x_{|K}^* + r\|_\infty < \frac{\varepsilon}{2}$. Hence, $\|x_{|K}^*\|_\infty \leq \|x_{|K}^* + r\|_\infty + |-r| < \varepsilon$. Finally, we observe that $f - (x_{|K}^* + r)$ has a unique minimum on K iff $f - x_{|K}^*$ has a unique minimum on K . This concludes the proof. \square

If $(E, \|\cdot\|)$ is a Banach space and E^* is its topological dual, the space $X = (E^*, \text{Weak}^*)$ is a l.c.t space. It is well know that in this case we have that $X^* = E$ (See for instance [Corollary 224., [7]]). In this case, the exposed points of a subset of X are called weak* exposed points and the closure of a subset coincides with its weak* closure. Recall that a Banach space E is said to be a Gâteaux differentiability space (GDS) if each convex continuous real valued function defined on E is Gâteaux differentiable at each point of a dense subset. The following corollary reveals a difference between a Gâteaux differentiability space and a general Banach space.

Corollary 1. *Let E be a Banach space and $f : E \rightarrow \mathbb{R}$, be a convex continuous function. Suppose that $\overline{\text{dom}(f^*)}^{\text{Weak}^*}$ is weak* compact metrizable of E^* (where f^* is the Fenchel conjugate of f). Then, f is Gâteaux differentiable at each point of a set $D \subset E$ which is "dense" in the following sens:*

$$\forall x \in E, \forall \varepsilon > 0; \exists y \in D / \sup_{x^* \in \overline{\text{dom}(f^*)}^{\text{Weak}^*}} |x^*(x - y)| < \varepsilon.$$

Proof. Set $K = \overline{\text{dom}(f^*)}^{\text{Weak}^*}$, which is a convex weak* compact metrizable subset of the l.c.t space $X = (E^*, \text{Weak}^*)$. Let $x \in X^* = E$ and $\hat{x} : x^* \mapsto x^*(x)$ for all $x^* \in E^*$. Since $f^* + \hat{x} : K \rightarrow \mathbb{R} \cup \{+\infty\}$ is proper lower semi continuous in (E^*, Weak^*) , we can apply Theorem 3 to obtain, for each $\varepsilon > 0$, there exists $x_\varepsilon \in E$ such that

- (1) $\sup_{x^* \in K} |x^*(x)| < \varepsilon$ and
- (2) $f^* + (\hat{x} - \hat{x}_\varepsilon)$ has a strict minimum on K , equivalently $f^* + (\hat{x} - \hat{x}_\varepsilon)$ has a strict minimum on E^* since $\text{dom}(f^*) \subset K$.

This shows that f is Gâteaux differentiable at $y := x - x_\varepsilon$ with Gâteaux derivative equal to x^* , by the classical Asplund-Rockafellar result since f is convex continuous (See for instance [1]). □

Let C be a non-empty subset of E^* . We denote by σ_C the support function defined on E by

$$\sigma_C(x) = \sup_{x^* \in C} x^*(x); \quad \forall x \in E.$$

Let $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semi continuous convex function. The inf convolution of f and σ_C is defined by

$$f \nabla \sigma_C(x) := \inf_{y \in E} \{f(x - y) + \sigma_C(y)\}.$$

From the above corollary we get that if K is a convex weak* compact metrizable subset of E^* and f is proper lower semi continuous convex function on E , then $f \nabla \sigma_K$ is Gâteaux differentiable at each point of a set $D \subset E$ which is "dense" in the sens given in Corollary 1. This gives, in any Banach space, a class of convex continuous functions which have points of Gâteaux differentiability.

3 Proof of the main result.

In [9], Phelps proved the following result.

Theorem 2. ([Theorem 6.2., [9] p. 95]) *A Banach space E is a GDS if and only if every weak* compact convex subset of E^* is the weak* closed convex hull of its weak* exposed points.*

With the notation of the introduction, Theorem 2 means that $(E^*, \text{Weak}^*) \in \Upsilon$ if and only if (E^*, Weak^*) has the E.P.P. It follows from Theorem 2, that if E is not a GDS (for instance if $E = l^\infty$), then there exists a compact convex subset of $X = (E^*, \text{Weak}^*)$ which is not the closed convex hull of its exposed points. In fact, we know from [Example 1.21., [9], p. 13] that the continuous semi-norm defined on l^∞ for all $x \in l^\infty$ by

$$p(x) = \limsup_n |x_n|$$

is nowhere Gâteaux differentiable. So, from [Proposition 6.9., [9], p. 98] we get that the set

$$K_0 = \{x^* \in (l^\infty)^* : \langle x^*, x \rangle \leq \limsup_n |x_n|; x \in l^\infty\}$$

is a convex compact subset of $((l^\infty)^*, \text{Weak}^*)$ with $\text{Exp}(K_0) = \emptyset$, in particular the space $((l^\infty)^*, \text{Weak}^*)$ does not have the E.P.P. However $\text{Ext}(K_0) \neq \emptyset$ by the Krein-Milman theorem. This situation never occurs for convex compact metrizable sets as asserts the following result.

Theorem 3. *Let K be a convex compact metrizable subset of a l.c.t space X . Then, K is the closed convex hull of its exposed points*

$$K = \overline{\text{conv}}(\text{Exp}(K)).$$

Proof. Since K is convex and closed set, it is clear that $\overline{\text{conv}}(\text{Exp}(K)) \subset K$. Now, let us prove that $K = \overline{\text{conv}}(\text{Exp}(K))$. Suppose towards a contradiction that there exists $k_0 \in K \setminus \overline{\text{conv}}(\text{Exp}(K))$. By the Hahn-Banach theorem, there exists $x_0^* \in X^* \setminus \{0\}$ and $r \in \mathbb{R}$ such that

$$\sup\{x_0^*(k) : k \in \overline{\text{conv}}(\text{Exp}(K))\} < r < x_0^*(k_0).$$

From Theorem 1 (applied with $f = x_0^*$), we can find $x_1^* \in X^*$ which exposes K at some point $k_1 \in \text{Exp}(K)$ and such that $x_1^*|_K$ is close (with respect to the sup-norm) to $x_0^*|_K$ in $(\text{Aff}(K), \|\cdot\|_\infty)$. Hence, x_1^* satisfies

$$\sup\{x_1^*(k) : k \in \overline{\text{conv}}(\text{Exp}(K))\} < r < x_1^*(k_0). \quad (1)$$

On the other hand

$$x_1^*(k_0) \leq \sup\{x_1^*(k) : k \in K\} = x_1^*(k_1)$$

which is a contradiction with (1), since $k_1 \in \text{Exp}(K)$. □

Remark 2. *We can replace exposed points by strongly exposed points in Theorem 3, since this two notions coincides on compact metrizable space. Recall that a point x of a metric set (C, d) is said to be a strongly exposed point, if there exists some continuous linear functional $x^* \in X^*$ which attains its strong maximum over (C, d) at x .*

Let us introduce a slightly different concept from exposed points.

Definition 5. Let K be a convex compact subset of a l.c.t space X . We say that a point $x \in K$ is an affine exposed point of K , and write $x \in \text{AExp}(K)$, if there exists some affine continuous map $\tau \in \text{Aff}(K)$ which attains its strict maximum over K at x .

Clearly, $\text{Exp}(K) \subseteq \text{AExp}(K) \subseteq \text{Ext}(K)$. Thus, from Theorem 3 and the Krein-Milman theorem, we have for all convex compact metrizable subset K of a l.c.t space X ,

$$K = \overline{\text{conv}}(\text{Exp}(K)) = \overline{\text{conv}}(\text{AExp}(K)) = \overline{\text{conv}}(\text{Ext}(K)).$$

So, from Theorem 1 gives the following proposition.

Proposition 2. Let K be a convex compact metrizable subset of a l.c.t space X . Then,

$$K = \overline{\text{conv}}(\text{AExp}(K))$$

and the set of affine maps which affine expose K at some point, has a complement which can be covered by countably many d.c hypersurface in $(\text{Aff}(K), \|\cdot\|_\infty)$.

We also have the following corollary which applies in any Banach spaces (even if E is not a GDS).

Corollary 2. Let E be a Banach space.

(1) Let K be a convex weak* compact metrizable subset of E^* , then

$$K = \overline{\text{conv}}^{\text{Weak}^*}(\text{Exp}(K)).$$

(2) Let K be a convex weak compact metrizable subset of E , then

$$K = \overline{\text{conv}}^{\|\cdot\|}(\text{Exp}(K)).$$

Proof. Since (E^*, Weak^*) and (E, Weak) are l.c.t spaces, the part (1) is a direct consequence of Theorem 3. Also from Theorem 3 we get that $K = \overline{\text{conv}}^{\text{Weak}^*}(\text{Exp}(K))$. The part (2) follows then from Mazur's lemma on the coincidence of weak and norm closure for convex sets. □

Recall that the class Ξ consists on all l.c.t space in which every compact subset is metrizable. We obtain immediately from Theorem 3 the following corollary.

Corollary 3. Every space from the class Ξ , has the E.P.P.

Examples 1. we mention here a simple example of spaces from Ξ .

(1) Every Fréchet space has the E.P.P.

(2) Every convex closed and bounded subset of a Fréchet-Montel space is the closed convex hull of its (strongly) exposed points (in Fréchet-Montel space, any closed bounded set is compact metrizable).

A classical example of a Fréchet-Montel space is the space $C^\infty(\Omega)$ of smooth functions on an open set Ω in \mathbb{R}^n . For examples of not metrizable spaces which belongs to Ξ , we have for example,

Proposition 3. *Let E be a separable Banach space. Then (E^*, Weak^*) and (E, Weak) belongs to the class Ξ , but are not metrizable.*

Proof. It is well known that the whole spaces (E^*, Weak^*) and (E, Weak) are not metrizable. It is also well known that a Banach space E is separable iff every compact subset of (E^*, Weak^*) is metrizable. Thus, $(E^*, \text{Weak}^*) \in \Xi$. For the space (E, Weak) , let K be a weak compact subset of E . Since E is separable, then K is also separable. Now, consider K as a subset of E^{**} by the canonical embedding, we get that K is norm separable and weak* compact subset of E^{**} , which implies from [Lemma 2, [3]] that K is weak* metrizable in E^{**} . In other words, K is weak metrizable. Thus $(E, \text{Weak}) \in \Xi$. \square

Several others not trivial examples of spaces from Ξ can be found in [4].

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