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Limited operators and differentiability.

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Abstract. Given Banach spaces Y and X and a linear continuous operator $T : Y \rightarrow X$, we prove that T is a limited operator if and only if, for every convex continuous function $f : X \rightarrow \mathbb{R}$ and every point $y \in Y$, $f \circ T$ is Fréchet-differentiable at $y \in Y$ whenever f is Gâteaux-differentiable at $T(y) \in X$. Some consequences will be given.

Keyword, phrase: Limited operators, Gâteaux-differentiability, Fréchet-differentiability, convex functions, extreme points.

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1 Introduction.

A subset A of a Banach space X is called limited, if every weak* null sequence $(p_n)_n$ in X^* converges uniformly on A , that is,

$$\lim_{n \rightarrow +\infty} \sup_{x \in A} |\langle p_n, x \rangle| = 0.$$

We know that every relatively compact subset of X is limited, but the converse is false in general. A bounded linear operator $T : Y \rightarrow X$ between Banach spaces Y and X is called limited, if T takes the closed unit ball B_Y of Y to a limited subset of X . It is easy to see that $T : Y \rightarrow X$ is limited if and only if, the adjoint operator $T^* : X^* \rightarrow Y^*$ takes weak* null sequence to norm null sequence. For some useful properties of limited sets and limited operators we refer to [6], [8] and [1].

We know that in finite dimensional Banach space, the notion of Gâteaux and Fréchet differentiability coincide for convex continuous functions. In [7], Borwein and Fabian proved that a Banach space Y is infinite dimensional if and only if, there exists on Y functions f having points at which f is Gateaux but not Fréchet differentiable (see Theorem 2 below). They also pointed in the introduction of [7] the observation that if the sup-norm $\|\cdot\|_\infty$ on c_0 is Gâteaux-differentiable at some point, then it is Fréchet-differentiable there. We observe in this article, that this phenomenon is not just about the sup-norm but more generally, for each convex continuous function $f : c_0 \rightarrow \mathbb{R}$, if the Fenchel biconjugate $(f^*)^* : l^\infty \rightarrow \mathbb{R}$ is Gâteaux-differentiable at some point $a \in c_0$ which is in the interior of its domain, then f is Fréchet-differentiable at a . In fact,

this phenomenon is due, as we will prove it, to the fact that the canonical embedding $i : c_0 \rightarrow l^\infty$ is a limited operator (see corollary 1 and the comment just before). The goal of this paper, is to prove the following characterization of limited operators in terms of the coincidence of Gâteaux and Fréchet differentiability of convex continuous functions.

Theorem 1. *Let Y and X be two Banach spaces and $T : Y \rightarrow X$ be a continuous linear operator. Then, T is a limited operator if and only if, for all convex continuous function $f : X \rightarrow \mathbb{R}$ and all $y \in Y$, $f \circ T$ is Fréchet-differentiable at $y \in Y$ whenever f is Gâteaux-differentiable at $T(y) \in X$.*

Some consequences will be deduced from this theorem. In particular, we complete the result of Borwein and Fabian in [7] by adding the part (2) in Theorem 2.

A real valued function f on a Banach space will be called a PGNF-function if there exists a point at which f is Gâteaux but not Fréchet differentiable. A JN-sequence (due to Josefson-Nissenzweig theorem) is a sequence $(p_n)_n$ in a dual space Y^* that is weak* null and $\inf_n \|p_n\| > 0$.

Theorem 2. *Let Y be a Banach space. Then the following assertions are equivalent.*

- (1) *There exists on Y a convex continuous PGNF-function.*
- (2) *There exists a Banach space X and a linear continuous nonlimited operator $T : Y \rightarrow X$.*
- (3) *There exists a linear continuous noncompact operator $T : Y \rightarrow c_0$.*
- (4) *There exists on Y^* an equivalent dual norm which is not weak* Kadec.*
- (5) *There exists a JN-sequence in Y^* .*
- (6) *There exists on Y an equivalent PGNF-norm.*
- (7) *Y is infinite dimensional.*

2 Limited operators and differentiability.

The proof of Theorem 1 is juste the combinaison of Theorem 3 and Theorem 4 which are given after two lemmas. The proof of Theorem 2 is given after Theorem 4.

Recall that the domain of a function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$, is the set

$$\text{dom}(f) := \{x \in X / f(x) < +\infty\}.$$

For a function f with $\text{dom}(f) \neq \emptyset$, the Fenchel-Moreau transform of f is defined on the dual space for all $p \in X^*$ by

$$f^*(p) := \sup_{x \in X} \{\langle p, x \rangle - f(x)\}.$$

The second transform $(f^*)^*$ is defined on the bidual X^{**} by the same formula. We denote by f^{**} , the restriction of $(f^*)^*$ to X , where X is identified to a subspace of X^{**} by the canonical embedding. Recall that the Fenchel theorem state that $f = f^{**}$ if and only if f is convex lower semicontinuous on X .

Definition 1. Let X be a Banach space and $g : X^* \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function such that $\text{dom}(g) \neq \emptyset$.

(1) We say that g has a strong minimum at $p \in X^*$ if $g(p) = \inf_{q \in X^*} g(q)$ and $(p_n)_n$ norm converge to p whenever $g(p_n) \rightarrow g(p)$.

(2) We say that g has a weak*-strong minimum at $p \in X^*$ if $g(p) = \inf_{q \in X^*} g(q)$ and $(p_n)_n$ weak* converge to p whenever $g(p_n) \rightarrow g(p)$.

A strong (resp. weak*-strong) minimum is in particular unique.

Lemma 1. Let X be a Banach space and $(p_n)_n$ be a weak* null sequence in X^* . Then, there exists a sequence $(x_n)_n$ in the sphere S_X of X such that, for all $p \in \overline{\text{co}}^{w^*} \{p_n : n \in \mathbb{N}\} \setminus \{0\}$ there exists $n \in \mathbb{N}$ such that $\langle p, x_n \rangle \neq 0$.

Proof. Since $(p_n)_n$ weak* converge to 0 it is norm bounded by the Banach-Steinhaus theorem. Thus, by setting $K = \overline{\text{co}}^{w^*} \{p_n : n \in \mathbb{N}\}$, clearly K is a weak* compact subset of X^* . According to Haydon's theorem [[13], Theorem 3.3] the weak* compact convex set K is the norm closed convex hull of its extreme points whenever $\text{ex}(K)$ (the set of extreme points of K) is norm separable. By the Milman theorem [[15], p.9] $\text{ex}(K) \subset \overline{\{p_n : n \in \mathbb{N}\}}^{w^*} = \{p_n : n \in \mathbb{N}\} \cup \{0\}$ so that $\text{ex}(K)$ is norm separable and, hence, by Haydon's theorem, K itself is weak* compact, convex, and norm separable. It follows that there exists a sequence $(q_n)_n$ of K such that $K = \overline{\{q_n : n \in \mathbb{N}\}}^{\|\cdot\|}$. According to the Bishop-Phelps theorem [5], the set

$$D = \{r \in X^* \mid r \text{ attains its supremum on the sphere } S_X\}$$

is norm-dense in the dual X^* . Thus, for each $n \in \mathbb{N}$, there exists $r_n \in D$ such that $\|q_n - r_n\| < \frac{1}{n+1}$. For each $n \in \mathbb{N}$, let $x_n \in S_X$ be such that $\|r_n\| = \langle r_n, x_n \rangle$. We claim that the sequence $(x_n)_n$ satisfies our goal. Indeed, let $p \in K = \overline{\{q_n : n \in \mathbb{N}\}}$ and suppose that $\langle p, x_n \rangle = 0$, for all $n \in \mathbb{N}$. For all $k \in \mathbb{N}^*$, there exists q_{n_k} such that $\|p - q_{n_k}\| < \frac{1}{k}$ and so $\|p - r_{n_k}\| < \frac{1}{k} + \frac{1}{n_k}$. It follows that

$$\begin{aligned} \|r_{n_k}\| &= \langle r_{n_k}, x_{n_k} \rangle \\ &= \langle r_{n_k}, x_{n_k} \rangle - \langle p, x_{n_k} \rangle \\ &\leq \|p - r_{n_k}\| \\ &< \frac{1}{k} + \frac{1}{n_k}. \end{aligned}$$

Hence, for all $k \in \mathbb{N}^*$, $\|p\| \leq \|p - r_{n_k}\| + \|r_{n_k}\| < 2(\frac{1}{k} + \frac{1}{n_k})$, which implies that $p = 0$. \square

Remark 1. We can deduce from the above lemma an analogous for weak null sequence $(x_n)_n$ in a Banach space X . It suffices to consider $(x_n)_n$ as a sequence of the bidual X^{**} and use Lemma 1.

Lemma 2. Let X be a Banach space and $(p_n)_n$ be a weak* null sequence in X^* . Set $K = \overline{\text{co}}^{w^*} \{p_n : n \in \mathbb{N}\}$. Then, there exists a convex and weak* lower semicontinuous function $g : X^* \rightarrow \mathbb{R} \cup \{+\infty\}$ with a norm bounded domain $\text{dom}(g) = K$, such that

- (1) $g(p) > 0 = g(0) = \min_{X^*}(g)$ for all $p \in X^* \setminus \{0\}$.
- (2) $\lim_{n \rightarrow +\infty} g(p_n) = \min_{X^*}(g)$.

Proof. From Lemme 1, there exists a sequence $(x_n)_n$ of X such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$, satisfying, for all $p \in K \setminus \{0\}$ there exists $n \in \mathbb{N}$ such that $\langle p, x_n \rangle \neq 0$. Set

$$g(p) = \sum_{k \geq 0} 2^{-k} (\langle p, x_k \rangle)^2 + \delta_K,$$

where δ_K denotes the indicator function, which is equal to 0 on K and equal to $+\infty$ otherwise. The function g is well defined, non-negative convex and weak* lower semicontinuous with $\text{dom}(g) = K$. If $g(p) = 0$, then $\langle p, x_k \rangle = 0$ for all $k \in \mathbb{N}$ which implies that $p = 0$. Hence, g has a unique minimum at 0. On the other hand, since the series $\sum_{k \geq 0} 2^{-k} (\langle p, x_k \rangle)^2$ converge uniformly on the closed ball $B_{X^*} \supseteq K$ and since $(p_n)_n$ weak* converge to 0, it follows that $\lim_{n \rightarrow +\infty} g(p_n) = 0 = \min_{X^*}(g)$. \square

Remark 2. Analogously to the lemma 2, using Remark 1, for each weak nul sequence $(x_n)_n$ in a Banach space X , there exists a convex lower semicontinuous function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ with domain $\text{dom}(f) = \overline{\text{co}}^w \{x_n : n \in \mathbb{N}\} = \overline{\text{co}}^{\|\cdot\|} \{x_n : n \in \mathbb{N}\}$ (by Mazur's lemma) such that f has a unique minimum at 0 and $(x_n)_n$ minimize f .

The "if" part of Theorem 1 is given by the following theorem.

Theorem 3. Let Y and X be Banach spaces and let $T : Y \rightarrow X$ be a linear continuous operator. Suppose that $f \circ T$ is Fréchet-differentiable at $y \in Y$ whenever $f : X \rightarrow \mathbb{R}$ is convex continuous and $f \circ T$ is Gâteaux-differentiable at $y \in Y$. Then T is a limited operator.

Proof. Let $(p_n)_n$ be a weak* null sequence in X^* . We want to prove that $\|T^*(p_n)\|_{Y^*} \rightarrow 0$. This is trivial if $p_n = 0$ for all $n \in \mathbb{N}$. Suppose that there exists $n \in \mathbb{N}$ such that $p_n \neq 0$, in this case we have

$$K := \overline{\text{co}}^{w^*} \{p_n : n \in \mathbb{N}\} \not\supseteq \{0\}.$$

From Lemma 2, there exists a convex and weak* lower semicontinuous function g with a norm bounded domain K , such that

- (1) $g(p) > 0 = g(0) = \min_{X^*}(g)$ for all $p \in X^* \setminus \{0\}$.
- (2) $\lim_{n \rightarrow +\infty} g(p_n) = \min_{X^*}(g)$.

Thus, there exists a convex and Lipschitz continuous function $f : X \rightarrow \mathbb{R}$ such that $g = f^*$ (we can take $f = g|_X^*$). Moreover, the function f is Gâteaux-differentiable at 0 with derivative $\nabla f(0) = 0$, this is due to the fact that $f^* = g$ has a unique minimum at 0. Thus, from our hypothesis, $f \circ T$ is Fréchet-differentiable at 0 with Fréchet derivative equal to 0. Thus, $(f \circ T)^*$ has a strong minimum at 0 (see [Corollary 2. [2]]). Now, we prove that $(T^*(p_n))_n$ minimize $(f \circ T)^*$. Indeed, on one hand, we have $0 = \min_{X^*}(g) = -g^*(0) = -f^{**}(0) = -f(0)$. On the other hand we have

$$\begin{aligned} 0 = -f(0) &\leq \sup_{y \in Y} \{-f \circ T(y)\} &:= (f \circ T)^*(0) \\ &\leq \sup_{x \in X} \{-f(x)\} \\ &= f^*(0) \\ &= g(0) \\ &= 0. \end{aligned}$$

It follows that $(f \circ T)^*(0) = 0$. On the other hand

$$\begin{aligned}
0 = (f \circ T)^*(0) &\leq (f \circ T)^*(T^*(p_n)) := \sup_{y \in Y} \{\langle T^*(p_n), y \rangle - f \circ T(y)\} \\
&= \sup_{y \in Y} \{\langle p_n, T(y) \rangle - f(T(y))\} \\
&\leq \sup_{x \in X} \{\langle p_n, x \rangle - f(x)\} \\
&= f^*(p_n) \\
&= g(p_n).
\end{aligned}$$

Since, $g(p_n) \rightarrow 0$, it follows that $(f \circ T)^*(T^*(p_n)) \rightarrow 0 = (f \circ T)^*(0)$. In other words, the sequence $(T^*(p_n))_n$ minimize $(f \circ T)^*$. Since $(f \circ T)^*$ has a strong minimum at 0, it follows that $\|T^*(p_n)\|_{Y^*} \rightarrow 0$, which implies that T is a limited operator. \square

The "only if" part of Theorem 1 is given by the following theorem.

Theorem 4. *Let Y and X be two Banach spaces and $T : Y \rightarrow X$ be a limited operator. Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$, be a convex lower semicontinuous function and let $a \in Y$ such that $T(a)$ belongs to the interior of $\text{dom}(f)$. Then, $f \circ T$ is Fréchet-differentiable at $a \in Y$ with Fréchet-derivative $T^*(Q) \in Y^*$, whenever f is Gâteaux-differentiable at $T(a) \in X$ with Gâteaux-derivative $Q \in X^*$.*

Proof. Since f is convex lower semicontinuous and $T(a)$ is in the interior of $\text{dom}(f)$, there exists $r_a > 0$ and $L_a > 0$ such that f is L_a -Lipschitz continuous on the closed ball $B_X(T(a), r_a)$. It is well known that there exists a convex L_a -Lipschitz continuous function \tilde{f}_a on X such that $\tilde{f}_a = f$ on $B_X(T(a), r_a)$ (See for instance Lemma 2.31 [14]). It follows that $\tilde{f}_a \circ T = f \circ T$ on $B_Y(a, \frac{r_a}{\|T\|})$, since $T(B_X(a, \frac{r_a}{\|T\|}))$ is a subset of $B_X(T(a), r_a)$. Replacing f by $\frac{1}{L_a}\tilde{f}_a$, we can assume without loss of generality that f is convex 1-Lipschitz continuous on X . It follows that $\text{dom}(f^*) \subset B_{X^*}$.

Claim. The function $q \mapsto f^*(q) - \langle q, T(a) \rangle$ has a weak*-strong minimum at Q .

Proof of the claim. Let $\varepsilon_n \rightarrow 0^+$ and $(p_n)_n \subset B_{X^*}$ such that

$$f^*(p_n) - \langle p_n, T(a) \rangle < \inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a) \rangle\} + \varepsilon_n. \quad (1)$$

From (1) and the fact that $f(T(y)) = f^{**}(T(y))$ for all $y \in Y$, we get for all $n \in \mathbb{N}$ and all $x \in X$

$$\begin{aligned}
f(T(a)) - \langle p_n, T(a) \rangle &= - \inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a) \rangle\} - \langle p_n, T(a) \rangle \\
&< -f^*(p_n) + \varepsilon_n \\
&\leq f(x) - \langle p_n, x \rangle + \varepsilon_n
\end{aligned} \quad (2)$$

We claim that $(p_n)_n$ weak* converge to Q . Suppose that the contrary hold, there exist $h \in X$ such that $(\langle p_n, h \rangle)_n$ does not converge to $\langle Q, h \rangle$ in \mathbb{R} . So for some subsequence $(p_{n_k})_k$ and $r > 0$ we have, for all $k \in \mathbb{N}$

$$\langle p_{n_k}, h \rangle - \langle Q, h \rangle > r \quad (3)$$

Let $t_{n_k} = 2\varepsilon_{n_k}/r$, using (2) with $x = T(a) + t_{n_k}h$ and (3) we obtain,

$$\begin{aligned} f(T(a) + t_{n_k}h) - f(T(a)) - \langle Q, t_{n_k}h \rangle &\geq \langle p_{n_k}, t_{n_k}h \rangle - \langle Q, t_{n_k}h \rangle - \varepsilon_n \\ &> t_{n_k}r - \varepsilon_n \\ &= t_{n_k}r/2. \end{aligned}$$

Consequently, f is not Gâteaux-differentiable at $T(a)$ which is a contradiction. Hence, $(p_n)_n$ weak* converge to Q . Since f^* is weak* lower semicontinuous on X^* , using (1) we obtain that the function $q \mapsto f^*(q) - \langle q, T(a) \rangle$ has a minimum at $Q \in B_{X^*}$. This minimum is a weak*-strong minimum since each sequence $(p_n)_n$ that minimize $q \mapsto f^*(q) - \langle q, T(a) \rangle$, weak* converge to Q . This end the proof of the claim.

Now, suppose by contradiction that $T^*(Q)$ is not the the Fréchet derivative of $f \circ T$ at a . There exist $\varepsilon > 0$, $t_n \rightarrow 0^+$ and $h_n \in Y$, $\|h_n\|_Y = 1$ such that for all $n \in \mathbb{N}^*$,

$$f \circ T(a + t_n h_n) - f \circ T(a) - \langle T^*(Q), t_n h_n \rangle > \varepsilon t_n \quad (4)$$

Let $r_n = t_n/n$ for all $n \in \mathbb{N}^*$ and choose $p_n \in B_{X^*}$ such that

$$f^*(p_n) - \langle p_n, T(a + t_n h_n) \rangle < \inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a + t_n h_n) \rangle\} + r_n \quad (5)$$

From (5) we get

$$f^*(p_n) - \langle p_n, T(a) \rangle < \inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a) \rangle\} + 2t_n \|T\| + r_n$$

This implies that the sequence $(p_n)_n$ minimize the function $q \mapsto f^*(q) - \langle q, T(a) \rangle$. Since we proved in the claim that the function $q \mapsto f^*(q) - \langle q, T(a) \rangle$ has a weak*-strong minimum on B_{X^*} at Q , it follows that $(p_n)_n$ weak* converge to Q and so (since T is limited) we have

$$\|T^*(p_n - Q)\|_{Y^*} \rightarrow 0. \quad (6)$$

On the other hand, since $f(T(a + t_n h_n)) = f^{**}(T(a + t_n h_n)) = -\inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a + t_n h_n) \rangle\}$, using (5) we obtain for all $y \in Y$

$$\begin{aligned} f \circ T(a + t_n h_n) - \langle p_n, T(a + t_n h_n) \rangle &< -f^*(p_n) + r_n \\ &\leq f \circ T(y) - \langle p_n, T(y) \rangle + r_n. \end{aligned}$$

Replacing y by a in the above inequality we obtain

$$f \circ T(a + t_n h_n) - \langle p_n, T(t_n h_n) \rangle \leq f \circ T(a) + r_n. \quad (7)$$

Combining (4) and (7) we get

$$\begin{aligned}
\varepsilon &< \langle p_n, T(h_n) \rangle - \langle T^*(Q), h_n \rangle + r_n/t_n \\
&= \langle T^*(p_n), h_n \rangle - \langle T^*(Q), h_n \rangle + \frac{1}{n} \\
&\leq \|T^*(p_n - Q)\|_{Y^*} + \frac{1}{n}
\end{aligned}$$

which is a contradiction with (6). Thus $f \circ T$ is Fréchet-differentiable at a with Fréchet derivative $T^*(Q)$. \square

We give now the proof of Theorem 2.

Proof of Theorem 2. The equivalences (1) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6) \Leftrightarrow (7) are given in [7]. The part (5) \implies (2) is given by taking $X = Y$ and $T = I$ the identity map. Indeed, there exists a sequence $(p_n)_n$ that weak* converge to 0 but $\inf_n \|I^*(p_n)\| = \inf_n \|p_n\| > 0$. So I cannot be a limited operator. Now we prove (2) \implies (1). Indeed, from Theorem 1, since $T : Y \rightarrow X$ is a non limited operator, there exists a convex continuous function $f : X \rightarrow \mathbb{R}$ and a point $y \in Y$ such that f is Gâteaux-differentiable at $T(y) \in X$ but $f \circ T$ is not Fréchet-differentiable at y . So $f \circ T$ is Gâteaux but not Fréchet-differentiable at y . Hence, $f \circ T$ is a convex continuous PGNF-function on Y . \square

There exists a class of Banach spaces $(E, \|\cdot\|_E)$ such that the canonical embedding $i : E \rightarrow E^{**}$ is a limited operator. This class contain in particular the space c_0 and any closed subspace F of c_0 (This class is also stable by product and quotient. For more informations see [8]). In this setting, Theorem 4 gives immediately the following corollary.

Corollary 1. *Suppose that the canonical embedding $i : E \rightarrow E^{**}$ is a limited operator. Let $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous function. Suppose that $x \in E$ belongs to the interior of $\text{dom}((f^*)^*)$ and that $(f^*)^*$ is Gâteaux-differentiable at x (we use the identification $i(x) = x$), then f is Fréchet-differentiable at x .*

Proposition 1. *Let Y and X be Banach spaces and $T : Y \rightarrow X$ be a limited operator with a dense range. Let $f : X \rightarrow \mathbb{R}$ be a convex continuous function. Then $f \circ T$ is Gâteaux-differentiable at $a \in Y$ if and only if, $f \circ T$ is Fréchet-differentiable at $a \in Y$.*

Proof. Suppose that $f \circ T$ is Gâteaux-differentiable at $a \in Y$. It follows that f is Gâteaux-differentiable at $T(a)$ with respect to the direction $T(Y)$ which is dense in X . It follows (from a classical fact on locally Lipschitz continuous functions) that f is Gâteaux-differentiable at $T(a)$ on X . So by Theorem 4, $f \circ T$ is Fréchet-differentiable at $a \in Y$. The converse is always true. \square

We obtain the following corollary by combining Proposition 1 and a delicate result due to Zajicek (see [Theorem 2; [17]]), which say that in a separable Banach space, the set of the points where a convex continuous function is not Gâteaux-differentiable, can be covered by countably many *d.c* (that is, delta-convex) *hypersurface*. Recall that in a separable Banach space Y , each set A which can be covered by countably many *d.c*

hypersurface is σ -lower porous, also σ -directionally porous; in particular it is both Aronszajn (equivalent to Gauss) null and Γ -null. For details about this notions of small sets we refer to [18] and references therein. Note that a limited set in a separable Banach space is relatively compact [6].

Corollary 2. *Let Y be a separable Banach space, X be a Banach spaces and $T : Y \rightarrow X$ be a compact operator with a dense range. Let $f : X \rightarrow \mathbb{R}$, be a convex and continuous function. Then, the set of all points at which $f \circ T$ is not Fréchet-differentiable can be covered by countably many d.c hypersurface.*

References

- [1] K. T. Andrews, *Dunford-Pettis sets in the space of Bochner integrable functions*, Math. Ann. 241, (1979), 35-41.
- [2] E. Asplund and R. T. Rockafellar. *Gradients of convex functions*. Trans. Amer. Math. Soc. , 139:443-467, 1969
- [3] M. Bachir and G. Lancien, *On the composition of differentiable functions*, Canadian Bull. of Math., 46(4), 2003, 481-494.
- [4] M. Bachir, *A non convex analogue to Fenchel duality*, J. Funct. Anal. 181, (2001) 300-312.
- [5] E. Bishop and R. R. Phelps *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc. 67 (1961), 97-98 .
- [6] J. Bourgain and J. Diestel, *Limited operators and strict cosingularity*, Math. Nachr. 119 (1984) 55-58.
- [7] J. M. Borwein and M. Fabian, *On convex functions having points of Gâteaux-differentiability which are not points of Féchet-differentiability*. Can. J. Math. Vol.45 (6), 1993 pp. 1121-1134.
- [8] H. Carrión, P. Galindo, and M.L Lourenco *Banach spaces whose bounded sets are bounding in the bidual* Annales Academiae Scientiarum Fennicae Mathematica Volumen 31, 2006, 61-70.
- [9] R. Deville and G. Godefroy and V. Zizler, *A smooth variational principle with applications to Hamilton-Jacobi equations in infinite dimensions*, J. Funct. Anal. 111, (1993) 197-212.
- [10] R. Deville and G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach Spaces*, Pitman Monographs No. 64, London: Longman, 1993.
- [11] Handbook of the Geometry of Banach spaces, Vol. 1. (eds., Johnson and Lindenstrauss), North Holland, 2001.
- [12] P. Habala, P. Hájek, V. Zizler, *Introduction to Banach Spaces*, Lect. Notes Math., Matfyzpress, Charles University, Prague, 1996.

- [13] R. Haydon, *An extreme point criterion for separability of a dual Banach space, and a new proof of a theorem of Corson*, Quart. J. Math. Oxford Ser. 27 (1976), 379-385.
- [14] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*. Lecture Notes in Mathematics 1364, (1993). Springer-Verlag, Berlin.
- [15] R. R. Phelps, *Lectures on Chaquet's theorem*, Van Nostrand, Princeton, N. J., 1966.
- [16] Pitt, H.R.: *A note on bilinear forms*. J. London Math. Soc. 11, 174-180 (1936)
- [17] L. Zajicek, *On the differentiation of convex functions in finite and infinite dimensional spaces*, Czechoslovak Math. J. 29(104) (1979), no. 3, 340-348.
- [18] L. Zajicek, *On sigma-porous sets in abstract spaces*, Abstract and Applied Analysis, vol. 2005, issue 5, pp. 509-534,