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Limited operators and differentiability.

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Abstract. We characterize the limited operators by differentiability of convex continuous functions. Given Banach spaces $Y$ and $X$ and a linear continuous operator $T : Y \rightarrow X$, we prove that $T$ is a limited operator if and only if, for every convex continuous function $f : X \rightarrow \mathbb{R}$ and every point $y \in Y$, $f \circ T$ is Fréchet differentiable at $y \in Y$ whenever $f$ is Gâteaux differentiable at $T(y) \in X$.

Keyword, phrase: Limited operators, Gâteaux differentiability, Fréchet differentiability, convex functions, extreme points.

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1 Introduction.

A subset $A$ of a Banach space $X$ is called limited, if every weak$^*$ null sequence $(p_n)_n$ in $X^*$ converges uniformly on $A$, that is,

$$\lim_{n \to +\infty} \sup_{x \in A} |\langle p_n, x \rangle| = 0.$$  

We know that every relatively compact subset of $X$ is limited, but the converse is false in general. A bounded linear operator $T : Y \rightarrow X$ between Banach spaces $Y$ and $X$ is called limited, if $T$ takes the closed unit ball $B_Y$ of $Y$ to a limited subset of $X$. It is easy to see that $T : Y \rightarrow X$ is limited if and only if, the adjoint operator $T^* : X^* \rightarrow Y^*$ takes weak$^*$ null sequence to norm null sequence. For useful properties of limited sets and limited operators we refer to [11], [4], [6] and [1].

We know that in a finite dimensional Banach space, the notions of Gâteaux and Fréchet differentiability coincide for convex continuous functions. In [5], Borwein and Fabian proved that a Banach space $Y$ is infinite dimensional if and only if, there exists on $Y$ functions $f$ having points at which $f$ is Gâteaux but not Fréchet differentiable. They also pointed in the introduction of [5] the observation that if the sup-norm $\|\cdot\|_{\infty}$ on $c_0$ is Gâteaux differentiable at some point, then it is Fréchet differentiable there. In this article we observe that this phenomenon is not just related to the sup-norm but more generally, for each convex lower semicontinuous function $g : l^\infty \rightarrow \mathbb{R} \cup \{+\infty\}$, if $g$ is Gâteaux differentiable at some point $a \in c_0$ which is in the interior of its domain, then the restriction of $g$ to $c_0$ is Fréchet differentiable at $a$. This hold in particular when $g = (f^*)^*$ is the Fenchel biconjugate of a convex continuous function $f : c_0 \rightarrow \mathbb{R}$. In fact, this phenomenon is due, (see Corollary 1 in the Appendix and the comment just before), to the fact that the canonical embedding $i : c_0 \rightarrow l^\infty$ is a limited operator (see the reference [6]).
The goal of this paper is to prove the following characterization of limited operators in terms of the coincidence of Gâteaux and Fréchet differentiability of convex continuous functions.

**Theorem 1.** Let $Y$ and $X$ be two Banach spaces and $T : Y \rightarrow X$ be a continuous linear operator. Then, $T$ is a limited operator if and only if, for every convex continuous function $f : X \rightarrow \mathbb{R}$ and every $y \in Y$, the function $f \circ T$ is Fréchet differentiable at $y \in Y$ whenever $f$ is Gâteaux differentiable at $T(y) \in X$.

As consequence we give, in Theorem 2 below, new characterizations of infinite dimensional Banach spaces, complementing a result of Borwein and Fabian in [[5], Theorem 1].

A real valued function $f$ on a Banach space will be called a PGNF-function (see [5]) if there exists a point at which $f$ is Gâteaux but not Fréchet differentiable. A JN-sequence (due to Josefson-Nissenzweig theorem, see [[7], Chapter XII]) is a sequence $(p_n)_{n \in \mathbb{N}}$ in a dual space $Y^*$ that is weak$^*$ null and $\inf_n \|p_n\| > 0$. We say that a function $g$ on $X^*$ has a norm-strong minimum (resp. weak$^*$-strong minimum) at $p \in X^*$ if $g(p) = \inf_{q \in X^*} g(q)$ and $(p_n)_{n \in \mathbb{N}}$ norm converges (resp. weak$^*$ converges) to $p$ whenever $g(p_n) \rightarrow g(p)$. A norm-strong minimum and weak$^*$-strong minimum are in particular unique.

**Theorem 2.** Let $Y$ be a Banach space. Then the following assertions are equivalent.

1. $Y$ is infinite dimensional.
2. There exists a JN-sequence in $Y^*$.
3. There exists a convex norm separable and weak$^*$ compact metrizable subset $K$ of $Y^*$ containing 0 and a continuous seminorm $h$ on $X^*$ which is weak$^*$ lower semicontinuous and weak$^*$ sequentially continuous, such that the restriction $h|_K$ has a weak$^*$-strong minimum but not norm-strong minimum at 0.
4. There exists a Banach space $X$ and a linear continuous non-limited operator $T : Y \rightarrow X$.
5. There exists on $Y$ a convex continuous PGNF-function.

In Section 2 we give some preliminary results, specially the key Lemma 2. In Section 3, we give the proof of Theorem 1 (divided into two part, Theorem 3 and Theorem 4) and the proof of Theorem 2. In Section 4 we give some complementary remarks.

## 2 Preliminaries.

We recall the following classical result.

**Lemma 1.** Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set $K$ such that
1. $K$ is Hausdorff with respect to $\mathcal{T}_1$,
2. $K$ is compact with respect to $\mathcal{T}_2$,
3. $\mathcal{T}_1 \subset \mathcal{T}_2$.

Then $\mathcal{T}_1 = \mathcal{T}_2$.

**Proof.** Let $F \subset K$ be a $\mathcal{T}_2$-closed set. It follows that $F$ is $\mathcal{T}_2$-compact, since $K$ is $\mathcal{T}_2$-compact. Let $\{O_i : i \in I\}$ be any cover of $F$ by $\mathcal{T}_1$-open sets. Since $\mathcal{T}_1 \subset \mathcal{T}_2$, then each of these sets is also $\mathcal{T}_2$-open. Hence, there exist a finite subcollection that covers $F$. It follows that $F$ is $\mathcal{T}_1$-compact and therefore is $\mathcal{T}_1$-closed since $\mathcal{T}_1$ is Hausdorff. This implies that $\mathcal{T}_2 \subset \mathcal{T}_1$. Consequently, $\mathcal{T}_1 = \mathcal{T}_2$. $
$

Now, we establish the following useful lemma. If $B$ is a subset of a dual Banach space $X^*$, we denote by $\overline{\text{co}}^w(B)$ the weak$^*$ closed convex hull of $B$. 

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Lemma 2. Let $X$ be a Banach space and $K$ be a subset of $X^*$.  

(1) Suppose that $K$ is norm separable, then there exists a sequence $(x_n)_n$ in the unit sphere $S_X$ of $X$ which separate the points of $K$ i.e. for all $p, p' \in K$, if $\langle p, x_n \rangle = \langle p', x_n \rangle$ then $p = p'$. Consequently, if $K$ is a weak$^*$ compact and norm separable set of $X^*$, then the weak$^*$ topology of $X^*$ restricted to $K$ is metrizable.

(2) Let $(p_n)_n$ be a weak$^*$ null sequence in $X^*$. Then, the set $\overline{\sigma^{w^*}}\{p_n : n \in \mathbb{N}\}$ is convex weak$^*$ compact and norm separable.

Proof. (1) Since $K$ is norm separable, then $K = K := \{a - b/(a, b) \in K \times K\}$ is also norm separable and so there exists a sequence $(q_n)_n$ of $K = K$ which is dense in $K - K$. According to the Bishop-Phelps theorem [3], the set

$$D = \{r \in X^* \mid r \text{ attains its supremum on the sphere } S_X\}$$

is norm-dense in the dual $X^*$. Thus, for each $n \in \mathbb{N}$, there exists $r_n \in D$ such that $\|q_n - r_n\| < \frac{1}{n} \|q_n\|$. For each $n \in \mathbb{N}$, let $x_n \in S_X$ be such that $\|r_n\| = \langle r_n, x_n \rangle$. We claim that the sequence $(x_n)_n$ separate the points of $K$. Indeed, let $q \in K - K$ and suppose that $\langle q, x_n \rangle = 0$, for all $n \in \mathbb{N}$. There exists a subsequence $(n_k)_k \subseteq K - K$ such that $\|q_{n_k} - q\| < \frac{1}{k}$ for all $k \in \mathbb{N}^*$ and so we have $\|r_{n_k} - q\| < \frac{1}{1 + n_k} + \frac{1}{k}$. It follows that

$$\|r_{n_k}\| = \langle r_{n_k}, x_{n_k} \rangle$$

$$= \langle r_{n_k}, x_{n_k} \rangle - \langle q, x_{n_k} \rangle$$

$$\leq \|r_{n_k} - q\|$$

$$< \frac{1}{1 + n_k} + \frac{1}{k}.$$

Hence, for all $k \in \mathbb{N}^*$, $\|q\| \leq \|q - r_{n_k}\| + \|r_{n_k}\| < 2\left(\frac{1}{1 + n_k} + \frac{1}{k}\right)$, which implies that $q = 0$, and so that $(x_n)_n$ separate the points of $K$. Now, suppose that $K$ is weak$^*$ compact subset of $X^*$. We show that the weak$^*$ topology of $X^*$ restricted to $K$ is metrizable. Indeed, each $x \in X$ determines a seminorm $\nu_x$ on $X^*$ given by

$$\nu_x(p) = |\langle p, x \rangle|, \quad p \in X^*.$$ 

The family of seminorms $(\nu_x)_{x \in X}$ induces the weak$^*$ topology $\sigma(X^*, X)$ on $X^*$. The subfamily $(\nu_{x_n})_n$ also induces a topology on $X^*$, which we will call $\mathcal{T}$. Since this is a smaller family of seminorms, we have $\mathcal{T} \subseteq \sigma(X^*, X)$. Suppose that $p, p' \in K$ and $\nu_{x_n}(p - p') = 0$ for all $n \in \mathbb{N}$. Then we have $\langle p, x_n \rangle = \langle p', x_n \rangle$ for all $n \in \mathbb{N}$ and so we have that $p = p'$ since $(x_n)_n$ separates the points of $K$. Consequently, $K$ is Hausdorff with respect to the topology $\mathcal{T}_{|K}$ (the restriction of $\mathcal{T}$ to $K$). Thus $\mathcal{T}_{|K}$ is a Hausdorff topology on $K$ induced from a countable family of seminorms, so this topology is metrizable. More precisely, $\mathcal{T}_{|K}$ is induced from the metric

$$d(p, p') := \sum_{n=0}^{+\infty} 2^{-n} \frac{\nu_{x_n}(p - p')}{1 + \nu_{x_n}(p - p')}.$$ 

Then we have that $K$ is Hausdorff with respect to $\mathcal{T}_{|K}$, and is compact with respect to $\sigma(X^*, X)_{|K}$. Lemma 1 implies that $\mathcal{T}_{|K} = \sigma(X^*, X)_{|K}$. Hence $\sigma(X^*, X)_{|K}$ is metrizable.

(2) Let $(p_n)_n$ be a weak$^*$ null sequence in $X^*$ and set $K = \overline{\sigma^{w^*}}\{p_n : n \in \mathbb{N}\}$. Clearly $K$ is a convex and weak$^*$ compact subset of $X^*$. According to Haydon’s theorem [8], Theorem 3.3] the weak$^*$ compact convex set $K$ is the norm closed convex hull of its extreme points whenever $\text{ex}(K)$ (the set of extreme points of $K$) is norm separable. By the Milman theorem [10], p.9 $\text{ex}(K) \subseteq \overline{\text{ue}}\{p_n : n \in \mathbb{N}\}$ so that $\text{ex}(K)$ is norm separable and, hence, by Haydon’s theorem, $K$ itself is weak$^*$ compact, convex, and norm separable. 

The following proposition will be used in the proof of Theorem 3.


Proposition 1. Let \( X \) be a Banach space and \( K \) be a weak* compact and norm separable subset of \( X^* \) containing 0. Then, there exists a continuous seminorm \( h \) on \( X^* \) satisfying

1. \( h \) is weak* lower semicontinuous and sequentially weak* continuous,
2. the restriction \( h|_K \) of \( h \) to \( K \) has a weak*-strong minimum at 0.

Proof. Using Lemma 2, there exists a sequence \( (x_k)_k \subset S_X \) which separate the points of \( K \). Define the function \( h : X^* \to \mathbb{R} \) as follows:

\[
h(x^*) = \left( \sum_{k \geq 0} 2^{-k}(x^*, x_k)^2 \right)^{\frac{1}{2}}, \quad \forall x^* \in X^*.
\]

It is clear that \( h \) is a seminorm, and since \( h(x^*) \leq \|x^*\| \) for all \( x^* \in X^* \), it is also continuous. Since \( h \) is the supremum of a sequence of weak* continuous functions, it is weak* lower semicontinuous. On the other hand, since the series \( \sum_{k \geq 0} 2^{-k}(x^*, x_k)^2 \) uniformly converges on bounded sets of \( X^* \) and since the maps \( \delta_k : x^* \mapsto (x^*, x_k) \) are weak* continuous for all \( k \in \mathbb{N} \), then \( h \) is sequentially weak* continuous. If \( p \in K \) and \( h(p) = 0 \), then \( \langle p, x_k \rangle = 0 \) for all \( k \in \mathbb{N} \) which implies that \( p = 0 \), since the sequence \( (x_k)_k \) separate the points of \( K \). Hence, the restriction of \( h \) to \( K \) has a unique minimum at 0. This minimum is necessarily a weak*-strong minimum since \( K \) is weak* metrizable by Lemma 2, this follows from a general fact which say that for every lower semicontinuous function on a compact metric space \((K, d)\), a unique minimum is necessarily a strong minimum for the metric \( d \) in question. \(\square\)

3 Limited operators and differentiability.

Recall that the domain of a function \( f : X \to \mathbb{R} \cup \{+\infty\} \), is the set

\[
dom(f) := \{x \in X / f(x) < +\infty\}.
\]

For a function \( f \) with \( \dom(f) \neq \emptyset \), the Fenchel transform of \( f \) is defined on the dual space for all \( p \in X^* \) by

\[
f^*(p) := \sup_{x \in X} \{ \langle p, x \rangle - f(x) \}.
\]

The second transform \((f^*)^*\) is defined on the bidual \( X^{**} \) by the same formula. We denote by \( f^{**} \), the restriction of \((f^*)^*\) to \( X \), where \( X \) is identified to a subspace of \( X^{**} \) by the canonical embedding. Recall that the Fenchel theorem state that \( f = f^{**} \) if and only if \( f \) is convex lower semicontinuous on \( X \).

The "if" part of Theorem 1 is given by the following theorem.

Theorem 3. Let \( Y \) and \( X \) be Banach spaces and let \( T : Y \to X \) be a linear continuous operator. Suppose that \( f \circ T \) is Fréchet differentiable at \( y \in Y \) whenever \( f : X \to \mathbb{R} \) is convex continuous and Gâteaux differentiable at \( T(y) \in Y \). Then \( T \) is a limited operator.

Proof. Let \((p_n)_n\) be a weak* null sequence in \( X^* \). We want to prove that \( \|T^*(p_n)\|_{Y^*} \to 0 \). Set

\[
K = \overline{\text{co}} \{p_n : n \in \mathbb{N}\}.
\]

According to Lemma 2, \( K \) is convex weak* compact and norm separable. Using Proposition 1, there exists a continuous seminorm which is weak* lower semicontinuous and sequentially weak* continuous \( h : X^* \to \mathbb{R} \) such that the restriction \( h|_K \) of \( h \) to \( K \) has a weak*-strong minimum at 0 and in particular \( \min_K h = h(0) = 0 \). Since the sequence \((p_n)_n\) weak* converges to 0, it follows that \( \lim_n h(p_n) = h(0) = \min_K h \). Thus, \((p_n)_n\) is a minimizing sequence for \( h|_K \). Set \( g = h + \delta_K \), where \( \delta_K \) denotes the indicator function, which is equal to 0 on \( K \) and equal to \(+\infty\) otherwise. Since \( K \) is convex, weak*-closed and norm bounded, then \( g \) is a convex and
weak* lower semicontinuous function with a norm bounded domain $dom(g) = K$. Moreover we have,

1. $g(p) > 0 = g(0) = \min_{X^*}(g)$ for all $p \in X^* \setminus \{0\}$.
2. $\lim_{n \to +\infty} g(p_n) = \min_{X^*}(g)$.

Hence, there exists a convex and Lipschitz continuous function $f : X \to \mathbb{R}$ such that $g = f^*$ (we can take $f = g^*_{\|\cdot\|_\infty}$). The function $f$ is Gâteaux differentiable at 0 with Gâteaux derivative $\nabla f(0) = 0$, this is due to the fact that $f^*$ is a weak*-minimum at 0 (we can see [Corollary 1. [2]]). Thus, from our hypothesis, $f \circ T$ is Fréchet differentiable at 0 with Fréchet derivative equal to 0. It follows that $(f \circ T)^*$ has a norm-strong minimum at 0 (see [Corollary 2. [2]]). Now, we prove that $(T^*(p_n))^*_{|Y^*}$ is a minimizing sequence for $(f \circ T)^*$, which will implies that $\|T^*(p_n)^*\|_{Y^*} \to 0$. Indeed, on one hand, we have $0 = \min_{X^*}(g) = -g^*(0) = -f(0)$. On the other hand we have

\[
0 = -f(0) \leq \sup_{y \in Y} \{-f \circ T(y)\} := (f \circ T)^*(0) \\
\leq \sup_{x \in X} \{-f(x)\} = f^*(0) = g(0) = 0.
\]

It follows that $(f \circ T)^*(0) = 0$. Hence, since $(f \circ T)^*$ has a minimum at 0, we obtain

\[
0 = (f \circ T)^*(0) \leq (f \circ T)^*(T^*(p_n)) := \sup_{y \in Y} \{\langle T^*(p_n), y \rangle - f \circ T(y)\} = \sup_{y \in Y} \{\langle p_n, T(y) \rangle - f(T(y))\} \leq \sup_{x \in X} \{\langle p_n, x \rangle - f(x)\} = f^*(p_n) = g(p_n).
\]

Since, $g(p_n) \to 0$, it follows that $(f \circ T)^*(T^*(p_n)) \to 0 = (f \circ T)^*(0)$. In other words, $(T^*(p_n))^*_{|Y^*}$ is a minimizing sequence for $(f \circ T)^*$. Since $(f \circ T)^*$ has a norm-strong minimum at 0, we obtain that $\|T^*(p_n)^*\|_{Y^*} \to 0$, which implies that $T$ is a limited operator.

The "only if" part of Theorem 1 is given by the following theorem.

**Theorem 4.** Let $Y$ and $X$ be two Banach spaces and $T : Y \to X$ be a limited operator. Let $f : X \to \mathbb{R} \cup \{+\infty\}$, be a convex lower semicontinuous function and let $a \in Y$ such that $T(a)$ belongs to the interior of $dom(f)$. Then, $f \circ T$ is Fréchet differentiable at $a \in Y$ with Fréchet-derivative $T^*(Q) \in Y^*$, whenever $f$ is Gâteaux differentiable at $T(a) \in X$ with Gâteaux-derivative $Q \in X^*$.

**Proof.** Since $f$ is convex lower semicontinuous and $T(a)$ is in the interior of $dom(f)$, there exists $r_a > 0$ and $L_a > 0$ such that $f$ is $L_a$-Lipschitz continuous on the closed ball $B_X(T(a), r_a)$. It is well known that there exists a convex $L_a$-Lipschitz continuous function $\tilde{f}_a$ on $X$ such that $\tilde{f}_a = f$ on $B_X(T(a), r_a)$ (See for instance Lemma 2.31 [9]). It follows that $\tilde{f}_a \circ T = f \circ T$ on $B_Y(a, \frac{r_a}{L_a})$, since $T(B_X(a, \frac{r_a}{L_a}))$ is a subset of $B_X(T(a), r_a)$ (we can assume that $T \neq 0$).

Replacing $f$ by $\frac{1}{L_a} \tilde{f}_a$, we can assume without loss of generality that $f$ is convex 1-Lipschitz continuous on $X$. It follows that $dom(f^*) \subset B_{X^*}$. (the closed unit ball of $X^*$).
Claim. Suppose that $f$ is Gâteaux differentiable at $T(a) \in X$ with Gâteaux-derivative $Q \in X^*$, then the function $q \mapsto f^*(q) - \langle q, T(a) \rangle$ has a weak*-strong minimum on $B_{X^*}$ at $Q$.

Proof of the claim. See [Corollary 1. [2]].

Now, suppose by contradiction that $T^*(Q)$ is not the the Fréchet derivative of $f \circ T$ at $a$. There exist $\varepsilon > 0$, $t_n \rightarrow 0^+$ and $h_n \in Y$, $\|h_n\|_Y = 1$ such that for all $n \in \mathbb{N}^*$,

$$f \circ T(a + t_n h_n) - f \circ T(a) - \langle T^*(Q), t_n h_n \rangle > \varepsilon t_n. \quad (1)$$

Let $r_n = t_n/n$ for all $n \in \mathbb{N}^*$ and choose $p_n \in B_{X^*}$ such that

$$f^*(p_n) - \langle p_n, T(a + t_n h_n) \rangle < \inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a + t_n h_n) \rangle\} + r_n. \quad (2)$$

From (2) we get

$$f^*(p_n) - \langle p_n, T(a) \rangle < \inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a) \rangle\} + 2t_n \|T\| + r_n.$$

This implies that the sequence $(p_n)_n$ minimize the function $q \mapsto f^*(q) - \langle q, T(a) \rangle$ on $B_{X^*}$. Using the claim, the function $q \mapsto f^*(q) - \langle q, T(a) \rangle$ has a weak*-strong minimum on $B_{X^*}$ at $Q$, it follows that $(p_n)_n$ weak* converges to $Q$ and so (since $T$ is limited) we have

$$\|T^*(p_n - Q)\|_{Y^*} \rightarrow 0. \quad (3)$$

On the other hand, since $f(T(a + t_n h_n)) = f^*(T(a + t_n h_n)) = \inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a + t_n h_n) \rangle\}$, using (2) we obtain for all $y \in Y$

$$f \circ T(a + t_n h_n) - \langle p_n, T(a + t_n h_n) \rangle < -f^*(p_n) + r_n \leq f \circ T(y) - \langle p_n, T(y) \rangle + r_n.$$ 

Replacing $y$ by $a$ in the above inequality we obtain

$$f \circ T(a + t_n h_n) - \langle p_n, T(t_n h_n) \rangle \leq f \circ T(a) + r_n. \quad (4)$$

Combining (1) and (4) we get

$$\varepsilon \leq \langle p_n, T(h_n) \rangle - \langle T^*(Q), h_n \rangle + r_n/t_n$$

$$= \langle T^*(p_n), h_n \rangle - \langle T^*(Q), h_n \rangle + \frac{1}{n}$$

$$\leq \|T^*(p_n - Q)\|_{Y^*} + \frac{1}{n}$$

which is a contradiction with (3). Thus $f \circ T$ is Fréchet differentiable at $a$ with Fréchet derivative $T^*(Q)$.

Now, we give the proof of Theorem 2.

Proof of Theorem 2. (1) $\implies$ (2) is the deeper Josefson-Nissenzweig theorem [[7], Chapter XII].

(2) $\implies$ (1) is well known.

(2) $\implies$ (3) Let $(p_n)_n$ be a weak* null sequence in $Y^*$ such that $\inf_n \|p_n\| > 0$ and set $K = \overline{co}^{\ast \ast}\{p_n : n \in \mathbb{N}\}$. By Lemma 2, the set $K$ is convex norm separable and weak* compact.
Thus, there exists a JN-sequence in PGNF-function on $Y$ differentiable at $y \in Y$. Hence, there exists a JN-sequence.

(3) $\implies$ (2) Since 0 is not a norm-strong minimum for the restriction $h|_K$, there exists a sequence $(p_n)_n$ that minimize $h$ on $K$ but $\|p_n\| \to 0$. Since $h|_K$ has a weak*-strong minimum at 0, it follows that $(p_n)_n$ weak* converges to 0. Hence, $(p_n)_n$ weak* converges to 0 but $\|p_n\| \to 0$. Thus, there exists a JN-sequence in $Y^*$.

(2) $\implies$ (4) This part is given by taking $X = Y$ and $T = I$ the identity map. Indeed, there exists a sequence $(p_n)_n$ which weak* converges to 0 but $\inf_n \|I^*(p_n)\| = \inf_n \|p_n\| > 0$. So $I$ cannot be a limited operator.

(4) $\implies$ (5). Indeed, if there exists a Banach space $X$ and a non-limited operator $T : Y \to X$, by using Theorem 1, there exists a convex continuous function $f : X \to \mathbb{R}$ and a point $y \in Y$ such that $f$ is Gâteaux differentiable at $T(y) \in X$ but $f \circ T$ is not Fréchet differentiable at $y$. So $f \circ T$ is Gâteaux but not Fréchet differentiable at $y$. Hence, $f \circ T$ is a convex continuous PGNF-function on $Y$.

(5) $\implies$ (2) Let $f$ be a PGNF-function on $Y$. We can assume without loss of generality that $f$ is Gâteaux differentiable at 0 with Gâteaux-derivative equal to 0, but $f$ is not Fréchet differentiable at 0. It follows from classical duality result (see Corollary 1. and Corollary 2. in [2]) that $f^*$ has a weak*-strong minimum but not norm-strong minimum at 0. Since 0 is not a norm-strong minimum for $f^*$, there exists a sequence $(p_n)_n \in X^*$ minimizing $f^*$ such that $\|p_n\| \to 0$. On the other hand, since $f^*$ has a weak*-strong minimum at 0, and $(p_n)_n$ minimize $f^*$, we have that $(p_n)_n$ weak* converges to 0. Thus, $(p_n)_n$ weak* converges to 0 but $\|p_n\| \to 0$. Hence, there exists a JN-sequence. 

**Canonical construction of PGNF-function.** There exist different ways to build a PGNF-function in infinite dimensional Banach spaces. We can find examples of such constructions in [5]. We present below a different method for constructing a PGNF-function on a Banach space $X$ canonically from a JN-sequence. Given a JN-sequence $(p_n)_n \subset X^*$, we set $K = \overline{\text{co}}\{p_n : n \in \mathbb{N}\}$. Using Lemma 2, there exists a sequence $(x_n)_n \in SX$ which separates the points of $K$, and as in the proof of Proposition 1, there exist a continuous seminorm $h$ which is weak* lower semicontinuous and weak* sequentially continuous such that $h|_K$ has a weak*-strong minimum at 0. The function $h$ is given explicitly as follows

$$h(x^*) = \left( \sum_{n \geq 0} 2^{-n} ((x^*, x_n))^2 \right)^{\frac{1}{2}}, \quad \forall x^* \in X^*.$$ 

Since $(p_n)_n$ weak* converges to 0, it follows that $(p_n)$ is a minimizing sequence for $h|_K$. Since $(p_n)_n$ is a JN-sequence, it follows that 0 is not a norm-strong minimum for $h|_K$. Define the function $f$ by

$$f(x) = (h + \delta_K)^*(\hat{x}), \quad \forall x \in X,$$

where $\delta_K$ denotes the indicator function, which is equal to 0 on $K$ and equal to $+\infty$ otherwise and where for each $x \in X$, we denote by $\hat{x} \in X^{**}$ the linear map $x^* \mapsto (x^*, x)$ for all $x^* \in X^*$. Then $f$ is convex Lipschitz continuous, Gâteaux differentiable at 0 (since $h + \delta_K$ has a weak*-strong minimum) but not Fréchet differentiable at 0 (because 0 is not a norm-strong minimum for $h + \delta_K$).

**4 Appendix.**

There exists a class of Banach spaces $(E, \|\cdot\|_E)$ such that the canonical embedding $i : E \to E^{**}$ is a limited operator. This class contains in particular the space $c_0$ and any closed subspace
$F$ of $c_0$ (This class is also stable by product and quotient. For more information see [6]). In this setting, Theorem 4 gives immediately the following corollary.

**Corollary 1.** Suppose that the canonical embedding $i : E \to E^{**}$ is a limited operator. Let $g : E^{**} \to \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous function. Suppose that $x \in E$ belongs to the interior of $\text{dom}(g)$ and that $g$ is Gâteaux differentiable at $x \in E$ (we use the identification $i(x) = x$), then the restriction of $g$ to $E$ is Fréchet differentiable at $x$. In particular, if $f : E \to \mathbb{R} \cup \{+\infty\}$ is convex lower semicontinuous, $x \in E$ belongs to the interior of $\text{dom}((f^*)^*)$ and $(f^*)^*$ is Gâteaux differentiable at $x$, then $f$ is Fréchet differentiable at $x$.

We obtain the following corollary by combining Proposition 2 and a delicate result due to Zajicěk (see [Theorem 2; [12]]), which say that in a separable Banach space, the set of the points where a convex continuous function is not Gâteaux differentiable, can be covered by countably many $d.c$ (that is, delta-convex) hypersurface. Recall that in a separable Banach space $Y$, each set $A$ which can be covered by countably many $d.c$ hypersurface is $\sigma$-lower porous, also $\sigma$-directionally porous; in particular it is both Aronszajn (equivalent to Gauss) null and $\Gamma$-null. For details about this notions of small sets we refer to [13] and references therein. Note that a limited set in a separable Banach space is relatively compact [4].

**Proposition 2.** Let $Y$ and $X$ be Banach spaces and $T : Y \to X$ be a limited operator with a dense range. Let $f : X \to \mathbb{R}$ be a convex continuous function. Then $f \circ T$ is Gâteaux differentiable at $a \in Y$ if and only if, $f \circ T$ is Fréchet differentiable at $a \in Y$.

**Proof.** Suppose that $f \circ T$ is Gâteaux differentiable at $a \in Y$. It follows that $f$ is Gâteaux differentiable at $T(a)$ with respect to the direction $T(Y)$ which is dense in $X$. It follows (from a classical fact on locally Lipschitz continuous functions) that $f$ is Gâteaux differentiable at $T(a)$ on $X$. So by Theorem 4, $f \circ T$ is Fréchet differentiable at $a \in Y$. The converse is always true.

**Corollary 2.** Let $Y$ be a separable Banach space, $X$ be a Banach spaces and $T : Y \to X$ be a compact operator with a dense range. Let $f : X \to \mathbb{R}$ be a convex and continuous function. Then, the set of all points at which $f \circ T$ is not Fréchet differentiable can be covered by countably many $d.c$ hypersurface.

**References**


