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Limited operators and differentiability.

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Abstract. We characterize the limited operators by differentiability of convex continuous functions. Given Banach spaces $Y$ and $X$ and a linear continuous operator $T : Y \to X$, we prove that $T$ is a limited operator if and only if, for every convex continuous function $f : X \to \mathbb{R}$ and every point $y \in Y$, $f \circ T$ is Fréchet differentiable at $y \in Y$ whenever $f$ is Gâteaux differentiable at $T(y) \in X$.

Keyword, phrase: Limited operators, Gâteaux differentiability, Fréchet differentiability, convex functions, extreme points.

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1 Introduction.

A subset $A$ of a Banach space $X$ is called limited, if every weak$^*$ null sequence $(p_n)_n$ in $X^*$ converges uniformly on $A$, that is,

$$\lim_{n \to +\infty} \sup_{x \in A} |\langle p_n, x \rangle| = 0.$$ We know that every relatively compact subset of $X$ is limited, but the converse is false in general. A bounded linear operator $T : Y \to X$ between Banach spaces $Y$ and $X$ is called limited, if $T$ takes the closed unit ball $B_Y$ of $Y$ to a limited subset of $X$. It is easy to see that $T : Y \to X$ is limited if and only if, the adjoint operator $T^* : X^* \to Y^*$ takes weak$^*$ null sequence to norm null sequence. For useful properties of limited sets and limited operators we refer to [11], [4], [6] and [1].

We know that in a finite dimensional Banach space, the notions of Gâteaux and Fréchet differentiability coincide for convex continuous functions. In [5], Borwein and Fabian proved that a Banach space $Y$ is infinite dimensional if and only if, there exists on $Y$ functions $f$ having points at which $f$ is Gâteaux but not Fréchet differentiable. They also pointed in the introduction of [5] the observation that if the sup-norm $\| \cdot \|_{\infty}$ on $c_0$ is Gâteaux differentiable at some point, then it is Fréchet differentiable there. In this article we observe that this phenomenon is not just related to the sup-norm but more generally, for each convex lower semicontinuous function $g : l^\infty \to \mathbb{R} \cup \{+\infty\}$, if $g$ is Gâteaux differentiable at some point $a \in c_0$ which is in the interior of its domain, then the restriction of $g$ to $c_0$ is Fréchet differentiable at $a$. This hold in particular when $g = (f^*)^*$ is the Fenchel biconjugate of a convex continuous function $f : c_0 \to \mathbb{R}$. In fact, this phenomenon is due, (see Corollary 1 in the Appendix and the comment just before), to the fact that the canonical embedding $i : c_0 \to l^\infty$ is a limited operator (see the reference [6]).
The goal of this paper is to prove the following characterization of limited operators in terms of the coincidence of Gâteaux and Fréchet differentiability of convex continuous functions.

**Theorem 1.** Let $Y$ and $X$ be two Banach spaces and $T : Y \to X$ be a continuous linear operator. Then, $T$ is a limited operator if and only if, for every convex continuous function $f : X \to \mathbb{R}$ and every $y \in Y$, the function $f \circ T$ is Fréchet differentiable at $y \in Y$ whenever $f$ is Gâteaux differentiable at $T(y) \in X$.

As a consequence we give, in Theorem 2 below, new characterizations of infinite dimensional Banach spaces, complementing a result of Borwein and Fabian in [5], Theorem 1.

A real valued function $f$ on a Banach space will be called a PGNF-function (see [5]) if there exists a point at which $f$ is Gâteaux but not Fréchet differentiable. A JN-sequence (due to Josefson-Nissenzweig theorem, see [7], Chapter XII) is a sequence $(p_n)_{n}$ in a dual space $Y^*$ that is weak-$^*$ null and $\inf_n \| p_n \| > 0$. We say that a function $g$ on $X^*$ has a norm-strong minimum (resp. weak-$^*$-strong minimum) at $p \in X^*$ if $g(p) = \inf_{g \in X^*} g(q)$ and $(p_n)_{n}$ norm converges (resp. weak-$^*$ converges) to $p$ whenever $g(p_n) \to g(p)$. A norm-strong minimum and weak-$^*$-strong minimum are in particular unique.

**Theorem 2.** Let $Y$ be a Banach space. Then the following assertions are equivalent.

1. $Y$ is infinite dimensional.
2. There exists a JN-sequence in $Y^*$.
3. There exists a convex norm separable and weak-$^*$ compact metrizable subset $K$ of $Y^*$ containing 0 and a continuous seminorm $h$ on $X^*$ which is weak-$^*$ lower semicontinuous and weak-$^*$ sequentially continuous, such that the restriction $h|_K$ has a weak-$^*$-strong minimum but not norm-strong minimum at 0.
4. There exists a Banach space $X$ and a linear continuous non-limited operator $T : Y \to X$.
5. There exists on $Y$ a convex continuous PGNF-function.

In Section 2 we give some preliminary results, specially the key Lemma 2. In Section 3, we give the proof of Theorem 1 (divided into two part, Theorem 3 and Theorem 4) and the proof of Theorem 2. In Section 4 we give some complementary remarks.

## 2 Preliminaries.

We recall the following classical result.

**Lemma 1.** Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set $K$ such that

1. $K$ is Hausdorff with respect $\mathcal{T}_1$,
2. $K$ is compact with respect to $\mathcal{T}_2$,
3. $\mathcal{T}_1 \subset \mathcal{T}_2$.

Then $\mathcal{T}_1 = \mathcal{T}_2$.

**Proof.** Let $F \subset K$ be a $\mathcal{T}_2$-closed set. It follows that $F$ is $\mathcal{T}_2$-compact, since $K$ is $\mathcal{T}_2$-compact. Let $\{O_i : i \in I\}$ be any cover of $F$ by $\mathcal{T}_1$-open sets. Since $\mathcal{T}_1 \subset \mathcal{T}_2$, then each of these sets is also $\mathcal{T}_2$-open. Hence, there exist a finite subcollection that covers $F$. It follows that $F$ is $\mathcal{T}_1$-compact and therefore is $\mathcal{T}_1$-closed since $\mathcal{T}_1$ is Hausdorff. This implies that $\mathcal{T}_2 \subset \mathcal{T}_1$. Consequently, $\mathcal{T}_1 = \mathcal{T}_2$.

Now, we establish the following useful lemma. If $B$ is a subset of a dual Banach space $X^*$, we denote by $\text{co}^w(B)$ the weak-$^*$ closed convex hull of $B$. 2
Lemma 2. Let $X$ be a Banach space and $K$ be a subset of $X^*$. 

(1) Suppose that $K$ is norm separable, then there exists a sequence $(x_n)_n$ in the unit sphere $S_X$ of $X$ which separate the points of $K$ i.e., for all $p, p' \in K$, if $\langle p, x_n \rangle = \langle p', x_n \rangle$ for all $n \in \mathbb{N}$, then $p = p'$. Consequently, if $K$ is a weak$^*$ compact and norm separable set of $X^*$, then the weak$^*$ topology of $X^*$ restricted to $K$ is metrizable.

(2) Let $(p_n)_n$ be a weak$^*$ null sequence in $X^*$. Then, the set $\varpi^* \{p_n : n \in \mathbb{N} \}$ is convex weak$^*$ compact and norm separable.

Proof. (1) Since $K$ is norm separable, then $K - K := \{a - b | (a, b) \in K \times K \}$ is also norm separable and so there exists a sequence $(q_n)_n$ of $K - K$ which is dense in $K - K$. According to the Bishop-Phelps theorem [3], the set 

$$D = \{ r \in X^* \mid r \text{ attains its supremum on the sphere } S_X \}$$

is norm-dense in the dual $X^*$. Thus, for each $n \in \mathbb{N}$, there exists $r_n \in D$ such that $\| q_n - r_n \| < \frac{1}{1 + n_k}$. For each $n \in \mathbb{N}$, let $x_n \in S_X$ be such that $\|r_n\| = \langle r_n, x_n \rangle$. We claim that the sequence $(x_n)_n$ separate the points of $K$. Indeed, let $q \in K - K$ and suppose that $\langle q, x_n \rangle = 0$, for all $n \in \mathbb{N}$.

There exists a subsequence $(q_{n_k})_k \subset K - K$ such that $\| q_{n_k} - q \| < \frac{1}{k}$ for all $k \in \mathbb{N}^*$ and so we have $\| r_{n_k} - q \| < \frac{1}{1 + n_k} + \frac{1}{k}$. It follows that 

$$\| r_{n_k} \| = \langle r_{n_k}, x_{n_k} \rangle = \langle r_{n_k}, x_{n_k} \rangle - \langle q, x_{n_k} \rangle \leq \| r_{n_k} - q \| < \frac{1}{1 + n_k} + \frac{1}{k}.$$ 

Hence, for all $k \in \mathbb{N}^*$, $\| q \| \leq \| q - r_{n_k} \| + \| r_{n_k} \| < 2(\frac{1}{1 + n_k} + \frac{1}{k})$, which implies that $q = 0$, and so that $(x_n)_n$ separate the points of $K$. Now, suppose that $K$ is weak$^*$ compact subset of $X^*$. We show that the weak$^*$ topology of $X^*$ restricted to $K$ is metrizable. Indeed, each $x \in X$ determines a seminorm $\nu_x$ on $X^*$ given by 

$$\nu_x(p) = |\langle p, x \rangle|, \quad p \in X^*.$$ 

The family of seminorms $(\nu_x)_{x \in X}$ induces the weak$^*$ topology $\sigma(X^*, X)$ on $X^*$. The subfamily $(\nu_{x_n})_n$ also induces a topology on $X^*$, which we will call $\mathcal{T}$. Since this is a smaller family of seminorms, we have $\mathcal{T} \subseteq \sigma(X^*, X)$. Suppose that $p, p' \in K$ and $\nu_{x_n}(p - p') = 0$ for all $n \in \mathbb{N}$. Then we have $\langle p, x_n \rangle = \langle p', x_n \rangle$ for all $n \in \mathbb{N}$ and so we have that $p = p'$ since $(x_n)_n$ separate the points of $K$. Consequently, $K$ is Hausdorff with respect to the topology $\mathcal{T}_K$ (the restriction of $\mathcal{T}$ to $K$). Thus $\mathcal{T}_K$ is a Hausdorff topology on $K$ induced from a countable family of seminorms, so this topology is metrizable. More precisely, $\mathcal{T}_K$ is induced from the metric 

$$d(p, p') := \sum_{n=0}^{+\infty} 2^{-n} \frac{\nu_{x_n}(p - p')}{1 + \nu_{x_n}(p - p')}.$$ 

Then we have that $K$ is Hausdorff with respect to $\mathcal{T}_K$, and is compact with respect to $\sigma(X^*, X)|_K$. Lemma 1 implies that $\mathcal{T}_K = \sigma(X^*, X)|_K$. Hence $\sigma(X^*, X)|_K$ is metrizable.

(2) Let $(p_n)_n$ be a weak$^*$ null sequence in $X^*$ and set $K = \varpi^* \{p_n : n \in \mathbb{N} \}$. Clearly $K$ is a convex and weak$^*$ compact subset of $X^*$. According to Haydon’s theorem [8], Theorem 3.3] the weak$^*$ compact convex set $K$ is the norm closed convex hull of its extreme points whenever $ex(K)$ (the set of extreme points of $K$) is norm separable. By the Milman theorem [10], p.9 $ex(K) \subset \{p_n : n \in \mathbb{N} \}^*$ = $\{p_n : n \in \mathbb{N} \} \cup \{0 \}$ so that $ex(K)$ is norm separable and, hence, by Haydon’s theorem, $K$ itself is weak$^*$ compact, convex, and norm separable. \[\square\]

The following proposition will be used in the proof of Theorem 3.
**Proposition 1.** Let $X$ be a Banach space and $K$ be a weak-$^*$ compact and norm separable subset of $X^*$ containing 0. Then, there exists a continuous seminorm $h$ on $X^*$ satisfying

1. $h$ is weak-$^*$ lower semicontinuous and sequentially weak-$^*$ continuous,
2. the restriction $h|_K$ of $h$ to $K$ has a weak-$^*$-strong minimum at 0.

**Proof.** Using Lemma 2, there exists a sequence $(x_k)_k \subset S_X$ which separate the points of $K$. Define the function $h : X^* \rightarrow \mathbb{R}$ as follows:

$$h(x^*) = \left( \sum_{k \geq 0} 2^{-k}(\langle x^*, x_k \rangle)^2 \right)^{\frac{1}{2}}, \quad \forall x^* \in X^*.$$ 

It is clear that $h$ is a seminorm, and since $h(x^*) \leq \|x^*\|$ for all $x^* \in X^*$, it is also continuous. Since $h$ is the supremum of a sequence of weak-$^*$ continuous functions, it is weak-$^*$ lower semicontinuous. On the other hand, since the series $\sum_{k \geq 0} 2^{-k}(\langle x^*, x_k \rangle)^2$ uniformly converges on bounded sets of $X^*$ and since the maps $\hat{x}_k : x^* \mapsto \langle x^*, x_k \rangle$ are weak-$^*$ continuous for all $k \in \mathbb{N}$, then $h$ is sequentially weak-$^*$ continuous. If $p \in K$ and $h(p) = 0$, then $(p, x_k) = 0$ for all $k \in \mathbb{N}$ which implies that $p = 0$, since the sequence $(x_k)_k$ separate the points of $K$. Hence, the restriction of $h$ to $K$ has a unique minimum at 0. This minimum is necessarily a weak-$^*$-strong minimum since $K$ is weak-$^*$ metrizable by Lemma 2, this follows from a general fact which say that for every lower semicontinuous function on a compact metric space $(K, d)$, a unique minimum is necessarily a strong minimum for the metric $d$ in question. \qedsymbol

## 3 Limited operators and differentiability.

Recall that the domain of a function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$, is the set

$$\text{dom}(f) := \{x \in X / f(x) < +\infty\}.$$ 

For a function $f$ with $\text{dom}(f) \neq \emptyset$, the Fenchel transform of $f$ is defined on the dual space for all $p \in X^*$ by

$$f^*(p) := \sup_{x \in X} \{\langle p, x \rangle - f(x)\}.$$ 

The second transform $(f^*)^*$ is defined on the bidual $X^{**}$ by the same formula. We denote by $f^{**}$, the restriction of $(f^*)^*$ to $X$, where $X$ is identified to a subspace of $X^{**}$ by the canonical embedding. Recall that the Fenchel theorem state that $f = f^{**}$ if and only if $f$ is convex lower semicontinuous on $X$.

The "if" part of Theorem 1 is given by the following theorem.

**Theorem 3.** Let $Y$ and $X$ be Banach spaces and let $T : Y \rightarrow X$ be a linear continuous operator. Suppose that $f \circ T$ is Fréchet differentiable at $y \in Y$ whenever $f : X \rightarrow \mathbb{R}$ is convex continuous and Gâteaux differentiable at $T(y) \in Y$. Then $T$ is a limited operator.

**Proof.** Let $(p_n)_n$ be a weak-$^*$ null sequence in $X^*$. We want to prove that $\|T^*(p_n)\|_{Y^*} \rightarrow 0$. Set

$$K = \overline{\text{conv}} \{p_n : n \in \mathbb{N}\}.$$ 

According to Lemma 2, $K$ is convex weak-$^*$ compact and norm separable. Using Proposition 1, there exists a continuous seminorm which is weak-$^*$ lower semicontinuous and sequentially weak-$^*$ continuous $h : X^* \rightarrow \mathbb{R}$ such that the restriction $h|_K$ of $h$ to $K$ has a weak-$^*$-strong minimum at 0 and in particular $\min_K h = h(0) = 0$. Since the sequence $(p_n)_n$ weak-$^*$ converges to 0, it follows that $\lim_n h(p_n) = h(0) = \min_K h$. Thus, $(p_n)_n$ is a minimizing sequence for $h|_K$. Set $g = h + \delta_K$, where $\delta_K$ denotes the indicator function, which is equal to 0 on $K$ and equal to $+\infty$ otherwise. Since $K$ is convex, weak-$^*$-closed and norm bounded, then $g$ is a convex and
Proof. Since \( f \) is well known that there exists a convex and Lipschitz continuous function \( f : X \to \mathbb{R} \) such that \( f = f^* \) (we can take \( f = g_{X^*}(x) \)). The function \( f \) is Gâteaux differentiable at 0 with Gâteaux derivative \( \nabla f(0) = 0 \), this is due to the fact that \( f^* = g \) has a weak*-strong minimum at 0 (we can see [Corollary 1. [2]]). Thus, from our hypothesis, \( f \circ T^* \) is Fréchet differentiable at 0 with Fréchet derivative equal to 0. It follows that \( (f \circ T)^* \) has a norm-strong minimum at 0 (see [Corollary 2. [2]]). Now, we prove that \( (T^*(p_n))_n \) is a minimizing sequence for \( (f \circ T)^* \), which will implies that \( \|T^*(p_n)\|_{Y^*} \to 0 \). Indeed, on one hand, we have \( 0 = \min_{X^*} (g) = -g^*(0) = -f(0) \). On the other hand we have

\[
0 = -f(0) \leq \sup_{y \in Y^*} \{ -f \circ T(y) \} := (f \circ T)^*(0) \\
\leq \sup_{x \in X^*} \{ -f(x) \} \\
= f^*(0) \\
= g(0) \\
= 0.
\]

It follows that \( (f \circ T)^*(0) = 0 \). Hence, since \( (f \circ T)^* \) has a minimum at 0, we obtain

\[
0 = (f \circ T)^*(0) \leq (f \circ T)^*(T^*(p_n)) := \sup_{y \in Y^*} \{ \langle T^*(p_n), y \rangle - f \circ T(y) \} \\
\leq \sup_{x \in X^*} \{ \langle p_n, x \rangle - f(x) \} \\
= f^*(p_n) \\
= g(p_n).
\]

Since \( g(p_n) \to 0 \), it follows that \( (f \circ T)^*(T^*(p_n)) \to 0 = (f \circ T)^*(0) \). In other words, \( (T^*(p_n))_n \) is a minimizing sequence for \( (f \circ T)^* \). Since \( (f \circ T)^* \) has a norm-strong minimum at 0, we obtain that \( \|T^*(p_n)\|_{Y^*} \to 0 \), which implies that \( T \) is a limited operator. \( \square \)

The "only if" part of Theorem 1 is given by the following theorem.

**Theorem 4.** Let \( Y \) and \( X \) be two Banach spaces and \( T : Y \to X \) be a limited operator. Let \( f : X \to \mathbb{R} \cup \{+\infty\} \), be a convex lower semicontinuous function and let \( a \in Y \) such that \( T(a) \) belongs to the interior of \( \text{dom}(f) \). Then, \( f \circ T \) is Fréchet differentiable at \( a \in Y \) with Fréchet-derivative \( T^*(Q) \in Y^* \), whenever \( f \) is Gâteaux differentiable at \( T(a) \in X \) with Gâteaux-derivative \( Q \in X^* \).

**Proof.** Since \( f \) is convex lower semicontinuous and \( T(a) \) is in the interior of \( \text{dom}(f) \), there exists \( r_a > 0 \) and \( L_a > 0 \) such that \( f \) is \( L_a \)-Lipschitz continuous on the closed ball \( B_X(T(a), r_a) \). It is well known that there exists a convex \( L_a \)-Lipschitz continuous function \( \tilde{f}_a \) on \( X \) such that \( \tilde{f}_a = f \) on \( B_X(T(a), r_a) \) (See for instance Lemma 2.31 [9]). It follows that \( \tilde{f}_a \circ T = f \circ T \) on \( B_X(a, \frac{r_a}{\sqrt{n}}) \), since \( T(B_X(a, \frac{r_a}{\sqrt{n}})) \) is a subset of \( B_X(T(a), r_a) \) (we can assume that \( T \neq 0 \)). Replacing \( f \) by \( \frac{1}{n} \tilde{f}_a \), we can assume without loss of generality that \( f \) is convex 1-Lipschitz continuous on \( X \). It follows that \( \text{dom}(f^*) \subset B_X \). (the closed unit ball of \( X^* \)).
Claim. Suppose that \( f \) is Gâteaux differentiable at \( T(a) \in X \) with Gâteaux-derivative \( Q \in X^* \), then the function \( q \mapsto f^*(q) - \langle q, T(a) \rangle \) has a weak*-strong minimum on \( B_{X^*} \) at \( Q \).

Proof of Theorem 2. See [Corollary 1. [2]].

Now, suppose by contradiction that \( T^*(Q) \) is not the the Fréchet derivative of \( f \circ T \) at \( a \). There exist \( \varepsilon > 0, t_n \rightarrow 0^+ \) and \( h_n \in Y, \|h_n\|_Y = 1 \) such that for all \( n \in \mathbb{N}^* , \)

\[
f \circ T(a + t_nh_n) - f \circ T(a) - (T^*(Q), t_nh_n) > \varepsilon t_n. \tag{1}
\]

Let \( r_n = t_n/n \) for all \( n \in \mathbb{N}^* \) and choose \( p_n \in B_{X^*} \) such that

\[
f^*(p_n) - \langle p_n, T(a + t_nh_n) \rangle < \inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a + t_nh_n) \rangle \} + r_n. \tag{2}
\]

From (2) we get

\[
f^*(p_n) - \langle p_n, T(a) \rangle < \inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a) \rangle \} + 2t_n\|T\| + r_n.
\]

This implies that the sequence \( (p_n) \) minimize the function \( q \mapsto f^*(q) - \langle q, T(a) \rangle \) on \( B_{X^*} \). Using the claim, the function \( q \mapsto f^*(q) - \langle q, T(a) \rangle \) has a weak*-strong minimum on \( B_{X^*} \) at \( Q \), it follows that \( (p_n) \) weak* converges to \( Q \) and so (since \( T \) is limited) we have

\[
\|T^*(p_n - Q)\|_{Y^*} \rightarrow 0. \tag{3}
\]

On the other hand, since \( f(T(a + t_nh_n)) = f^*(T(a + t_nh_n)) = -\inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a + t_nh_n) \rangle \} \), using (2) we obtain for all \( y \in Y \)

\[
f \circ T(a + t_nh_n) - \langle p_n, T(a + t_nh_n) \rangle \leq -f^*(p_n) + r_n \leq f \circ T(y) - \langle p_n, T(y) \rangle + r_n.
\]

Replacing \( y \) by \( a \) in the above inequality we obtain

\[
f \circ T(a + t_nh_n) - \langle p_n, T(t_nh_n) \rangle \leq f \circ T(a) + r_n. \tag{4}
\]

Combining (1) and (4) we get

\[
\varepsilon < \langle p_n, T(h_n) \rangle - \langle T^*(Q), h_n \rangle + r_n/t_n
\]

\[
= \langle T^*(p_n), h_n \rangle - \langle T^*(Q), h_n \rangle + \frac{1}{n}
\]

\[
\leq \|T^*(p_n - Q)\|_{Y^*} + \frac{1}{n}
\]

which is a contradiction with (3). Thus \( f \circ T \) is Fréchet differentiable at \( a \) with Fréchet derivative \( T^*(Q) \).

Now, we give the proof of Theorem 2.

Proof of Theorem 2. (1) \( \Rightarrow \) (2) is the deeper Josefson-Nissenzweig theorem [[7], Chapter XII].

(2) \( \Rightarrow \) (1) is well known.

(2) \( \Rightarrow \) (3) Let \( (p_n) \) be a weak* null sequence in \( Y^* \) such that \( \inf_n \|p_n\| > 0 \) and set \( K = \overline{co}^{w^*} \{p_n : n \in \mathbb{N} \} \). By Lemma 2, the set \( K \) is convex norm separable and weak* compact.
metrizable. On the other hand, from Proposition 1, there exists a continuous seminorm $h$ which is weak* lower semicontinuous and weak* sequentially continuous on $Y^*$ such that the restriction of $h$ to $K$ has a weak*-strong minimum at $0$. It remains to show that $0$ is not a norm-strong minimum for $h|_K$. Indeed, since $(p_n)_n$ is weak* null and $h$ is weak* sequentially continuous, then $\lim_n h(p_n) = h(0) = \inf_K h$. So $(p_n)_n$ is a minimizing sequence for $h|_K$ which does not converge to $0$ since $\inf_n \|p_n\| > 0$. Hence, $0$ is not a norm-strong minimum for $h|_K$.

(3) $\implies$ (2) Since $0$ is not a norm-strong minimum for the restriction $h|_K$, there exists a sequence $(p_n)_n$ that minimize $h$ on $K$ but $\|p_n\| \not\to 0$. Since $h|_K$ has a weak*-strong minimum at $0$, it follows that $(p_n)_n$ weak* converges to $0$. Hence, $(p_n)_n$ weak* converges to $0$ but $\|p_n\| \not\to 0$. Thus, there exists a JN-sequence in $Y^*$.

(2) $\implies$ (4) This part is given by taking $X = Y$ and $T = I$ the identity map. Indeed, there exists a sequence $(p_n)_n$ which weak* converges to $0$ but $\inf_n \|f(p_n)\| = \inf_n \|p_n\| > 0$. So $I$ cannot be a limited operator.

(4) $\implies$ (5). Indeed, if there exists a Banach space $X$ and a non-limited operator $T : Y \to X$, by using Theorem 1, there exists a convex continuous function $f : X \to \mathbb{R}$ and a point $y \in Y$ such that $f$ is Gâteaux differentiable at $T(y) \in X$ but $f \circ T$ is not Fréchet differentiable at $y$. So $f \circ T$ is Gâteaux but not Fréchet differentiable at $y$. Hence, $f \circ T$ is a convex continuous PGNF-function on $Y$.

(5) $\implies$ (2) Let $f$ be a PGNF-function on $Y$. We can assume without loss of generality that $f$ is Gâteaux differentiable at $0$ with Gâteaux-derivative equal to $0$, but $f$ is not Fréchet differentiable at $0$. It follows from classical duality result (see Corollary 1. and Corollary 2. in [2]) that $f^*$ has a weak*-strong minimum but not norm-strong minimum at $0$. Since $0$ is not a norm-strong minimum for $f^*$, there exists a sequence $(p_n)_n \in X^*$ minimizing $f^*$ such that $\|p_n\| \to 0$. On the other hand, since $f^*$ has a weak*-strong minimum at $0$, and $(p_n)_n$ minimize $f^*$, we have that $(p_n)_n$ weak* converges to $0$. Thus, $(p_n)_n$ weak* converges to $0$ but $\|p_n\| \not\to 0$. Hence, there exists a JN-sequence. $\square$

**Canonical construction of PGNF-function.** There exist different way to build a PGNF-function in infinite dimensional Banach spaces. We can find examples of such constructions in [5]. We present below a different method for constructing a PGNF-function on a Banach space $X$ canonically from a JN-sequence. Given a JN-sequence $(p_n)_n \subset X^*$, we set $K = \overline{\{p_n : n \in \mathbb{N}\}}$. Using Lemma 2, there exists a sequence $(x_n)_n \in \mathcal{S}_X$ which separates the points of $K$, and as in the proof of Proposition 1, there exist a continuous seminorm $h$ which is weak* lower semicontinuous and weak* sequentially continuous such that $h|_K$ has a weak*-strong minimum at $0$. The function $h$ is given explicitly as follows

$$h(x^*) = \left( \sum_{n \geq 0} 2^{-n} \langle x^*, x_n \rangle \right)^{\frac{1}{2}}, \quad \forall x^* \in X^*.$$ 

Since $(p_n)_n$ weak* converges to $0$, it follows that $(p_n)$ is a minimizing sequence for $h|_K$. Since $(p_n)_n$ is a JN-sequence, it follows that $0$ is not a norm-strong minimum for $h|_K$. Define the function $f$ by

$$f(x) = (h + \delta_K)^*(\hat{x}), \quad \forall x \in X,$$

where $\delta_K$ denotes the indicator function, which is equal to $0$ on $K$ and equal to $+\infty$ otherwise and where for each $x \in X$, we denote by $\hat{x} \in X^{**}$ the linear map $x^* \mapsto (x^*, x)$ for all $x^* \in X^*$. Then $f$ is convex Lipschitz continuous, Gâteaux differentiable at $0$ (since $h + \delta_K$ has a weak*-strong minimum) but not Fréchet differentiable at $0$ (because $0$ is not a norm-strong minimum for $h + \delta_K$).

4 Appendix.

There exists a class of Banach spaces $(E, \|\cdot\|_E)$ such that the canonical embedding $i : E \to E^{**}$ is a limited operator. This class contains in particular the space $c_0$ and any closed subspace
Corollary 1. Suppose that the canonical embedding $i : E \rightarrow E^{**}$ is a limited operator. Let $g : E^{**} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous function. Suppose that $x \in E$ belongs to the interior of $\text{dom}(g)$ and that $g$ is Gâteaux differentiable at $x \in E$ (we use the identification $i(x) = x$), then the restriction of $g$ to $E$ is Fréchet differentiable at $x$. In particular, if $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex lower semicontinuous function, $x \in E$ belongs to the interior of $\text{dom}((f^*)^*)$ and $(f^*)^*$ is Gâteaux differentiable at $x$, then $f$ is Fréchet differentiable at $x$.

We obtain the following corollary by combining Proposition 2 and a delicate result due to Zajíček (see [Theorem 2; [12]]), which say that in a separable Banach space, the set of the points where a convex continuous function is not Gâteaux differentiable, can be covered by countably many $d.c$ (that is, delta-convex) hypersurface. Recall that in a separable Banach space $Y$, each set $A$ which can be covered by countably many $d.c$ hypersurface is $\sigma$-lower porous, also $\sigma$-directionally porous; in particular it is both Aronszajn (equivalent to Gauss) null and $\Gamma$-null. For details about this notions of small sets we refer to [13] and references therein. Note that a limited set in a separable Banach space is relatively compact [4].

Proposition 2. Let $Y$ and $X$ be Banach spaces and $T : Y \rightarrow X$ be a limited operator with a dense range. Let $f : X \rightarrow \mathbb{R}$ be a convex continuous function. Then $f \circ T$ is Gâteaux differentiable at $a \in Y$ if and only if, $f \circ T$ is Fréchet differentiable at $a \in Y$.

Proof. Suppose that $f \circ T$ is Gâteaux differentiable at $a \in Y$. It follows that $f$ is Gâteaux differentiable at $T(a)$ with respect to the direction $T(Y)$ which is dense in $X$. It follows (from a classical fact on locally Lipschitz continuous functions) that $f$ is Gâteaux differentiable at $T(a)$ on $X$. So by Theorem 4, $f \circ T$ is Fréchet differentiable at $a \in Y$. The converse is always true.

Corollary 2. Let $Y$ be a separable Banach space, $X$ be a Banach spaces and $T : Y \rightarrow X$ be a compact operator with a dense range. Let $f : X \rightarrow \mathbb{R}$, be a convex and continuous function. Then, the set of all points at which $f \circ T$ is not Fréchet differentiable can be covered by countably many $d.c$ hypersurface.

References


