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A useful lemma for Lagrange multiplier rules in infinite dimension.

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Abstract. We give some reasonable and usable conditions on a sequence of norm one in a dual Banach space under which the sequence does not converge to the origin in the \(w^\ast\)-topology. These requirements help to ensure that the Lagrange multipliers are nontrivial, when we are interested for example on the infinite dimensional infinite-horizon Pontryagin Principles for discrete-time problems.

Key word, phrase: Baire category theorem, Subadditive and continuous map, Multiplier rules.

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1 Introduction.

Let \(Z\) be a Banach space and \(Z^\ast\) its topological dual. It is well known that in infinite dimensional separable Banach space, it is always true that the origin in \(Z^\ast\) is the \(w^\ast\)-limit of a sequence from the unit sphere \(S_2\) as it is in its \(w^\ast\)-closure. In this paper, we look about reasonable and usable conditions on a sequence of norm one in \(Z^\ast\) such that this sequence does not converge to the origin in the \(w^\ast\)-topology. This situation has the interest, when we are looking for a nontrivial Lagrange multiplier for optimization problems, and was encountered several times in the literature. See for example [1] and [3]. To guarantee that the multiplier are nontrivial at the limit, the authors in [3] used the following lemma from [[2], pp. 142, 135].

Definition 1. A subset \(Q\) of a Banach space \(Z\) is said to be of finite codimension in \(Z\) if there exists a point \(z_0\) in the closed convex hull of \(Q\) such that the closed vector space generated by \(Q - z_0 := \{q - z_0| q \in Q\}\) is of finite codimension in \(Z\) and the closed convex hull of \(Q - z_0\) has a nonempty interior in this vector space.
Lemma 1. ([2], pp. 142, 135) Let $Q \subset Z$ be a subset of finite codimension in $Z$. Let $(f_k)_k \subset Z^*$ and $\epsilon_k \geq 0$ and $\epsilon_k \to 0$ such that

i) $\|f_k\| \geq \delta > 0$, for all $k \in \mathbb{N}$ and $f_k \rightharpoonup f$.

ii) for all $z \in Q$, and for all $k \in \mathbb{N}$, $f_k(z) \geq -\epsilon_k$.

Then, $f \neq 0$.

Note that, this is not the most general situation. Indeed, one can meet as in [1], a situation where the part ii) of the above lemma is not uniform on $z \in Z$, and depends on other parameter as follows: for all $z \in \overline{co}(Q)$, there exists $C_z \in \mathbb{R}$ such that for all $k \in \mathbb{N}$, $f_k(z) \geq -\epsilon_k C_z$. The principal Lemma 4 that we propose in this paper, will permit to include this very useful situation. This lemma is based on the Baire category theorem.

2 Preliminary Lemmas.

We need the following classical lemma. We denote by $Int(A)$ the topological interior of a set $A$.

Lemma 2. Let $C$ be a convex subset of a normed vector space. Let $x_0 \in Int(C)$ and $x_1 \in \overline{C}$. Then, for all $\alpha \in [0,1]$, we have $\alpha x_0 + (1 - \alpha)x_1 \in Int(C)$.

We deduce the following lemma.

Lemma 3. Let $(F, \|\cdot\|_F)$ be a Banach space and $C$ be a closed convex subset of $F$ with non empty interior. Suppose that $D \subset C$ is a closed subset of $C$ with no empty interior in $(C, \|\cdot\|_F)$ (for the topology induced by $C$). Then, the interior of $D$ is non empty in $(F, \|\cdot\|_F)$.

Proof. On one hand, there exists $x_0$ such that $x_0 \in Int(C)$. On the other hand, since $D$ has no empty interior in $(C, \|\cdot\|_F)$, there exists $x_1 \in D$ and $\epsilon_1 > 0$ such that $B_F(x_1, \epsilon_1) \cap C \subset D$. By using Lemma 2, $\forall \alpha \in [0,1]$, we have $\alpha x_0 + (1 - \alpha)x_1 \in Int(C)$. Since $\alpha x_0 + (1 - \alpha)x_1 \to x_1$ when $\alpha \to 0$, then there exist some small $\alpha_0$ and an integer number $N \in \mathbb{N}^*$ such that $B_F(\alpha_0 x_0 + (1 - \alpha_0)x_1, \frac{1}{N}) \subset B(x_1, \epsilon_1) \cap C \subset D$. Thus $D$ has a non empty interior in $F$. \qed

3 The principal Lemma.

We give now our principal lemma. We denote by $\overline{co}(X)$ the closed convex hull of $X$.

Lemma 4. Let $Z$ be a Banach space. Let $(p_n)_n$ be a sequence of subadditive and continuous map on $Z$ and $(\lambda_n)_n \subset \mathbb{R}^+$ be a sequence of nonnegative real number such that $\lambda_n \to 0$. Let $A$ be a non empty subset of $Z$, $a \in \overline{co}(A)$ and $F := \overline{\text{span}}(A-a)$ the closed vector space generated by $A$. Suppose that $\overline{co}(A-a)$ has no empty interior in $F$ and that

(1) for all $z \in \overline{co}(A)$, there exists $C_z \in \mathbb{R}$ such that for all $n \in \mathbb{N}$:

$$p_n(z) \leq C_z \lambda_n.$$

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(2) for all $z \in F$, $\limsup_n p_n(z) \leq 0$.

Then, for all bounded subset $B$ of $F$, we have

$$\limsup_n \left( \sup_{h \in B} p_n(h) \right) \leq 0.$$  

**Proof.** For each $m \in \mathbb{N}$, we set

$$F_m := \{ z \in \overline{\omega}(A) : p_n(z) \leq m\lambda_n, \forall n \in \mathbb{N} \}.$$  

The sets $F_m$ are closed subsets of $Z$. Indeed,

$$F_m = \left( \bigcap_{n \in \mathbb{N}} p_n^{-1}([-\infty, m\lambda_n]) \right) \cap (\overline{\omega}(A))$$

where, for each $n \in \mathbb{N}$, $p_n^{-1}([-\infty, m\lambda_n])$ is a closed subset of $Z$ by the continuity of $p_n$. On the other hand, we have $\overline{\omega}(A) = \bigcup_{m \in \mathbb{N}} F_m$. Indeed, let $z \in \overline{\omega}(A)$, there exists $C_z \in \mathbb{R}$ such that $p_n(z) \leq C_z\lambda_n$ for all $n \in \mathbb{N}$. If $C_z \leq 0$, then $z \in F_0$. If $C_z > 0$, it suffices to take $m_1 := \left\lfloor C_z \right\rfloor + 1$ where $\left\lfloor C_z \right\rfloor$ denotes the floor of $C_z$ to have that $z \in F_{m_1}$.

We deduce then that $F_m - a$ are closed and that $\overline{\omega}(A) - a = \bigcup_{m \in \mathbb{N}} (F_m - a)$. Using the Baire Theorem on the complete metric space $(\overline{\omega}(A) - a, \| \cdot \|_F)$, we get an $m_0 \in \mathbb{N}$ such that $F_{m_0} - a$ has no empty interior in $(\overline{\omega}(A) - a, \| \cdot \|_F)$. Since by hypothesis $\overline{\omega}(A) - a$ has no empty interior in $F$, using Lemma 3 to obtain that $F_{m_0} - a$ has no empty interior in $F$. So there exists $z_0 \in F_{m_0} - a$ and some integer number $N \in \mathbb{N}$ such that $B_{F}(z_0, \frac{1}{N}) \subset F_{m_0} - a$. In other words, for all $z \in B_{F}(b, \frac{1}{N}) \subset F_{m_0}$ (with $b := a + z_0 \in F_{m_0} \subset F$) and all $n \in \mathbb{N}$, we have:

$$p_n(z) \leq m_0\lambda_n. \quad (1)$$

Let now $B$ a bounded subset of $F$, there exists an integer number $K_B \in \mathbb{N}$ such that $B \subset B_F(0, K_B)$. On the other hand, for all $h \in B$, there exists $z_h \in B_{F}(b, \frac{1}{N})$ such that $h = K_B.N(z_h - b)$. So using (1) and the subadditivity of $p_n$, we obtain that, for all $n \in \mathbb{N}$:

$$p_n(h) = p_n(K_B.N(z_h - b)) \leq K_B.N p_n(z_h - b) \leq K_B.N(p_n(z_h) + p_n(-b)) \leq K_B.Nm_0\lambda_n + K_B.Np_n(-b).$$

On passing to the supremum on $B$, we obtain for all $n \in \mathbb{N}$,

$$\sup_{h \in B} p_n(h) \leq K_B.Nm_0\lambda_n + K_B.Np_n(-b).$$

Since $\lambda_n \to 0$, we have

$$\limsup_n \left( \sup_{h \in B} p_n(h) \right) \leq K_B.N \limsup_n p_n(-b) \leq 0.$$  

This concludes the proof. \qed
As an immediate consequence, we obtain the following corollary.

**Corollary 1.** Let $Z$ be a Banach space. Let $(f_n)_n \subset Z^*$ be a sequence of linear and continuous functionals on $Z$ and let $(\lambda_n)_n \subset \mathbb{R}^+$ such that $\lambda_n \to 0$. Let $A$ be a nonempty subset of $Z$, $a \in \overline{co}(A)$ and $F := \text{span}(A-a)$ the closed vector space generated by $A$. Suppose that $\overline{co}(A-a) (= \overline{co}(A) - a)$ has no empty interior in $F$ and that

1. for all $z \in \overline{co}(A)$, there exists a real number $C_z$ such that, for all $n \in \mathbb{N}$, we have $f_n(z) \leq C_z \lambda_n$.

2. $f_n \overset{w^*}{\to} 0$.

Then, $\|(f_n)_{|F}\|_{F^*} \to 0$.

**Proof.** The proof follows Lemma 4 with the subadditive and continuous maps $f_n$ and the bounded set $B := S_{F^*}$. \hfill \Box

In the following corollary, the inequality in ii) depends on $z \in Z$ unlike in [2] where the inequality is uniformly independent on $z$. Note also that if $C_z$ does not depend on $z$, the condition ii) is also true by replacing: for all $z \in \overline{co}(Q)$ by for all $z \in Q$.

**Corollary 2.** Let $Q \subset Z$ be a subset of finite codimension in $Z$. Let $(f_k)_k \subset Z^*$ and $\epsilon_k \geq 0$ and $\epsilon_k \to 0$ such that

i) $\|f_k\| \geq \delta > 0$, for all $k \in \mathbb{N}$, and $f_k \overset{w^*}{\to} f$.

ii) for all $z \in \overline{co}(Q)$, there exists $C_z \in \mathbb{R}$ such that for all $k \in \mathbb{N}$, $f_k(z) \geq -\epsilon_k C_z$.

Then, $f \neq 0$.

**Proof.** Suppose by contradiction that $f = 0$. By applying Corollary 1 to $Q$ and $-f_k$, we obtain $\|(f_k)_{|F}\|_{F^*} \to 0$ where $F := \text{span}(Q - z_0)$. Since $F$ is of finite codimension in $Z$, there exists a finite-dimensional subspace $E$ of $Z$, such that $Z = F \oplus E$. Thus, there exists $L > 0$ such that

$$\|f_k\| \leq L \left( \|(f_k)_{|E}\|_{E^*} + \|(f_k)_{|F}\|_{F^*} \right).$$

Then, using i) we obtain $\lim_k \|(f_k)_{|E}\|_{E^*} \geq \frac{\delta}{L}$. Since the weak-star topology and the norm topology coincides on $E$ because of finite dimension, we have that $0 = \|(f)_{|E}\|_{E^*} = \lim_k \|(f_k)_{|E}\|_{E^*} \geq \frac{\delta}{L} > 0$, which is a contradiction. Hence $f \neq 0$. \hfill \Box

**References**

