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A useful lemma for Lagrange multiplier rules in infinite dimension.

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Abstract. We give some reasonable and usable conditions on a sequence of norm one in a dual Banach space under which the sequence does not converge to the origin in the \(w^*\)-topology. These requirements help to ensure that the Lagrange multipliers are nontrivial, when we are interested for example on the infinite dimensional infinite-horizon Pontryagin Principles for discrete-time problems.

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1 Introduction.

Let \(Z\) be a Banach space and \(Z^*\) its topological dual. It is well known that in infinite dimensional separable Banach space, it is always true that the origin in \(Z^*\) is the \(w^*\)-limit of a sequence from the unit sphere \(S_Z\), as it is in its \(w^*\)-closure. In this paper, we look about reasonable and usable conditions on a sequence of norm one in \(Z^*\) such that this sequence does not converge to the origin in the \(w^*\)-topology. This situation has the interest, when we are looking for a nontrivial Lagrange multiplier for optimization problems, and was encountered several times in the literature. See for example [1] and [3]. To guarantee that the multiplier are nontrivial at the limit, the authors in [3] used the following lemma from [2], pp. 142, 135.

Definition 1. A subset \(Q\) of a Banach space \(Z\) is said to be of finite codimension in \(Z\) if there exists a point \(z_0\) in the closed convex hull of \(Q\) such that the closed vector space generated by \(Q - z_0 := \{q - z_0 | q \in Q\}\) is of finite codimension in \(Z\) and the closed convex hull of \(Q - z_0\) has a no empty interior in this vector space.
Lemma 1. ([2], pp. 142, 135) Let \( Q \subset Z \) be a subset of finite codimension in \( Z \). Let \( (f_k)_k \subset Z^* \) and \( \epsilon_k \geq 0 \) and \( \epsilon_k \to 0 \) such that

1. \( \|f_k\| \geq \delta > 0 \), for all \( k \in \mathbb{N} \) and \( f_k \overset{w^*}{\to} f. \)
2. For all \( z \in Q \), and for all \( k \in \mathbb{N} \), \( f_k(z) \geq -\epsilon_k \).

Then, \( f \neq 0. \)

Note that, this is not the most general situation. Indeed, one can meet as in [1], a situation where the part ii) of the above lemma is not uniform on \( z \in Z \), and depends on other parameter as follows: for all \( z \in \overline{co}(Q) \), there exists \( C_z \in \mathbb{R} \) such that for all \( k \in \mathbb{N} \), \( f_k(z) \geq -\epsilon_k C_z \). The principal Lemma 4 that we propose in this paper, will permit to include this very useful situation. This lemma is based on the Baire category theorem.

2 Preliminary Lemmas.

We need the following classical lemma. We denote by \( Int(A) \) the topological interior of a set \( A \).

Lemma 2. Let \( C \) be a convex subset of a normed vector space. Let \( x_0 \in Int(C) \) and \( x_1 \in C \). Then, for all \( \alpha \in ]0,1] \), we have \( \alpha x_0 + (1 - \alpha)x_1 \in Int(C). \)

We deduce the following lemma.

Lemma 3. Let \( (F, \| \cdot \|_F) \) be a Banach space and \( C \) be a closed convex subset of \( F \) with non empty interior. Suppose that \( D \subset C \) is a closed subset of \( C \) with no empty interior in \( (C, \| \cdot \|_F) \) (for the topology induced by \( C \)). Then, the interior of \( D \) is non empty in \( (F, \| \cdot \|_F) \).

Proof. On one hand, there exists \( x_0 \) such that \( x_0 \in Int(C) \). On the other hand, since \( D \) has no empty interior in \( (C, \| \cdot \|_F) \), there exists \( x_1 \in D \) and \( \epsilon_1 > 0 \) such that \( B_F(x_1, \epsilon_1) \cap C \subset D \). By using Lemma 2, \( \forall \alpha \in ]0,1] \), we have \( \alpha x_0 + (1 - \alpha)x_1 \in Int(C) \). Since \( \alpha x_0 + (1 - \alpha)x_1 \to x_1 \) when \( \alpha \to 0 \), then there exist some small \( \alpha_0 \) and an integer number \( N \in \mathbb{N}^* \) such that \( B_F(\alpha_0 x_0 + (1 - \alpha_0)x_1, \frac{1}{N}) \subset B(x_1, \epsilon_1) \cap C \subset D \). Thus \( D \) has a non empty interior in \( F \). \( \square \)

3 The principal Lemma.

We give now our principal lemma. We denote by \( \overline{co}(X) \) the closed convex hull of \( X \).

Lemma 4. Let \( Z \) be a Banach space. Let \( (p_n)_n \) be a sequence of subadditive and continuous map on \( Z \) and \( (\lambda_n)_n \subset \mathbb{R}^+ \) be a sequence of nonnegative real number such that \( \lambda_n \to 0 \). Let \( A \) be a non empty subset of \( Z \), \( a \in \overline{co}(A) \) and \( F := \text{span}(A - a) \) the closed vector space generated by \( A \). Suppose that \( \overline{co}(A - a) \) has no empty interior in \( F \) and that

1. For all \( z \in \overline{co}(A) \), there exists \( C_z \in \mathbb{R} \) such that for all \( n \in \mathbb{N} \):
   \[
   p_n(z) \leq C_z \lambda_n.
   \]
(2) For all \( z \in F \), \( \limsup_n p_n(z) \leq 0 \).

Then, for all bounded subset \( B \) of \( F \), we have

\[
\limsup_n \left( \sup_{h \in B} p_n(h) \right) \leq 0.
\]

**Proof.** For each \( m \in \mathbb{N} \), we set

\[
F_m := \{ z \in \operatorname{co}(A) : p_n(z) \leq m\lambda_n, \forall n \in \mathbb{N} \}.
\]

The sets \( F_m \) are closed subsets of \( Z \). Indeed,

\[
F_m = \left( \bigcap_{n \in \mathbb{N}} p_n^{-1}([-\infty, m\lambda_n]) \right) \cap (\operatorname{co}(A))
\]

where, for each \( n \in \mathbb{N} \), \( p_n^{-1}([-\infty, m\lambda_n]) \) is a closed subset of \( Z \) by the continuity of \( p_n \). On the other hand, we have \( \overline{\operatorname{co}}(A) = \bigcup_{m \in \mathbb{N}} F_m \). Indeed, let \( z \in \overline{\operatorname{co}}(A) \), there exists \( C_z \in \mathbb{R} \) such that \( p_n(z) \leq C_z \lambda_n \) for all \( n \in \mathbb{N} \). If \( C_z \leq 0 \), then \( z \in F_0 \). If \( C_z > 0 \), it suffices to take \( m_1 := \lfloor C_z \rfloor + 1 \) where \( \lfloor C_z \rfloor \) denotes the floor of \( C_z \) to have that \( z \in F_{m_1} \). We deduce then that \( F_m - a \) are closed and that \( \overline{\operatorname{co}}(A) - a = \bigcup_{m \in \mathbb{N}} (F_m - a) \). Using the Baire Theorem on the complete metric space \( (\overline{\operatorname{co}}(A) - a, \|\cdot\|_F) \), we get an \( m_0 \in \mathbb{N} \) such that \( F_{m_0} - a \) has no empty interior in \( (\overline{\operatorname{co}}(A) - a, \|\cdot\|_F) \). Since by hypothesis \( \overline{\operatorname{co}}(A) - a \) has no empty interior in \( F \), using Lemma 3 to obtain that \( F_{m_0} - a \) has no empty interior in \( F \). So there exists \( z_0 \in F_{m_0} - a \) and some integer number \( N \in \mathbb{N}^* \) such that \( B_F(z_0, \frac{1}{N}) \subset F_{m_0} - a \). In other words, for all \( z \in B_F(b, \frac{1}{N}) \subset F_{m_0} \) (with \( b := a + z_0 \in F_{m_0} \subset F \)) and all \( n \in \mathbb{N} \), we have:

\[
p_n(z) \leq m_0\lambda_n.
\]

(1)

Let now \( B \) a bounded subset of \( F \), there exists an integer number \( K_B \in \mathbb{N}^* \) such that \( B \subset B_F(0, K_B) \). On the other hand, for all \( h \in B \), there exists \( z_h \in B_F(b, \frac{1}{N}) \) such that \( h = K_B.N(z_h - b) \). So using (1) and the subadditivity of \( p_n \), we obtain that, for all \( n \in \mathbb{N} \):

\[
p_n(h) = p_n(K_B.N(z_h - b)) \leq K_B.Np_n(z_h - b) \leq K_B.N(p_n(z_h) + p_n(-b)) \leq K_B.Nm_0\lambda_n + K_B.Np_n(-b).
\]

On passing to the supremum on \( B \), we obtain for all \( n \in \mathbb{N} \),

\[
\sup_{h \in B} p_n(h) \leq K_B.Nm_0\lambda_n + K_B.Np_n(-b).
\]

Since \( \lambda_n \to 0 \), we have

\[
\limsup_n \left( \sup_{h \in B} p_n(h) \right) \leq K_B.N \limsup_n p_n(-b) \leq 0.
\]

This concludes the proof. \( \Box \)
As an immediate consequence, we obtain the following corollary.

**Corollary 1.** Let $Z$ be a Banach space. Let $(f_n)_n \subset Z^*$ be a sequence of linear and continuous functionals on $Z$ and let $(\lambda_n)_n \subset \mathbb{R}^+$ such that $\lambda_n \to 0$. Let $A$ be a nonempty subset of $Z$, $a \in \overline{\text{co}}(A)$ and $F := \text{span}(A - a)$ the closed vector space generated by $A$. Suppose that $\overline{\text{co}}(A - a) (= \overline{\text{co}}(A) - a)$ has no empty interior in $F$ and that

$(1)$ for all $z \in \overline{\text{co}}(A)$, there exists a real number $C_z$ such that, for all $n \in \mathbb{N}$, we have $f_n(z) \leq C_z \lambda_n$.

$(2)$ $f_n w^* \to 0$.

Then, $\| (f_n)|_F \|_{F^*} \to 0$.

**Proof.** The proof follows Lemma 4 with the subadditive and continuous maps $f_n$ and the bounded set $B := S_{F^*}$. 

In the following corollary, the inequality in $ii)$ depends on $z \in Z$ unlike in [2] where the inequality is uniformly independent on $z$. Note also that if $C_z$ does not depend on $z$, the condition $ii)$ is also true by replacing; for all $z \in \overline{\text{co}}(Q)$ by for all $z \in Q$.

**Corollary 2.** Let $Q \subset Z$ be a subset of finite codimension in $Z$. Let $(f_k)_k \subset Z^*$ and $\epsilon_k \geq 0$ and $\epsilon_k \to 0$ such that

$i)$ $\| f_k \| \geq \delta > 0$, for all $k \in \mathbb{N}$, and $f_k w^* \to f$.

$ii)$ for all $z \in \overline{\text{co}}(Q)$, there exists $C_z \in \mathbb{R}$ such that for all $k \in \mathbb{N}$, $f_k(z) \geq -\epsilon_k C_z$.

Then, $f \neq 0$.

**Proof.** Suppose by contradiction that $f = 0$. By applying Corollary 1 to $Q$ and $-f_k$, we obtain $\| (f_k)|_F \|_{F^*} \to 0$ where $F := \text{span}(Q - z_0)$. Since $F$ is of finite codimension in $Z$, there exists a finite-dimensional subspace $E$ of $Z$, such that $Z = F \oplus E$. Thus, there exists $L > 0$ such that

$$\| f_k \| \leq L \left( \| (f_k)|_E \|_{E^*} + \| (f_k)|_F \|_{F^*} \right).$$

Then, using $i)$ we obtain $\lim_k \| (f_k)|_E \|_{E^*} \geq \frac{\delta}{L}$. Since the weak-star topology and the norm topology coincides on $E$ because of finite dimension, we have that $0 = \| (f)|_E \|_{E^*} = \lim_k \| (f_k)|_E \|_{E^*} \geq \frac{\delta}{L} > 0$, which is a contradiction. Hence $f \neq 0$. 

**References**

