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# AN EXTENSION OF BIFRACTIONAL BROWNIAN MOTION

XAVIER BARDINA\* AND KHALIFA ES-SEBAIY

ABSTRACT. In this paper we introduce and study a self-similar Gaussian process that is the bifractional Brownian motion  $B^{H,K}$  with parameters  $H \in (0, 1)$  and  $K \in (1, 2)$  such that  $HK \in (0, 1)$ . A remarkable difference between the case  $K \in (0, 1)$  and our situation is that this process is a semimartingale when  $2HK = 1$ .

## 1. Introduction

Houdré and Villa in [7] gave the first introduction to the bifractional Brownian motion (bifBm)  $B^{H,K} = (B_t^{H,K}; t \geq 0)$  with parameters  $H \in (0, 1)$  and  $K \in (0, 1]$  which is defined as a centered Gaussian process, with covariance function

$$R^{H,K}(t, s) = E \left( B_t^{H,K} B_s^{H,K} \right) = \frac{1}{2^K} \left( (t^{2H} + s^{2H})^K - |t - s|^{2HK} \right),$$

for every  $s, t \geq 0$ .

The case  $K = 1$  corresponds to the fractional Brownian motion (fBm) with Hurst parameter  $H$ . Some properties of the bifractional Brownian motion have been studied by Russo and Tudor in [12]. In fact, in [12] it is shown that the bifractional Brownian motion behaves as a fractional Brownian motion with Hurst parameter  $HK$ . A stochastic calculus with respect to this process has been recently developed by Kruk, Russo and Tudor [9] and Es-Sebaiy and Tudor [6].

In this paper we prove that, with  $H \in (0, 1)$  and  $HK \in (0, 1)$ , the process  $B^{H,K}$  can be extended for  $1 < K < 2$ . The case  $H = \frac{1}{2}$  and  $1 < K < 2$  plays a role to give an extension of sub-fractional Brownian motion (subfBm) (see [4]). The subfBm  $(\xi_t^h, t \geq 0)$  with parameter  $0 < h \leq 2$  is a centered Gaussian process with covariance:

$$E \left( \xi_t^h \xi_s^h \right) = C_h \left( t^{2h} + s^{2h} - \frac{1}{2} \left( (t+s)^{2h} + |t-s|^{2h} \right) \right); \quad s, t \geq 0$$

where  $C_h = 1$  if  $0 < h < 1$  and  $C_h = 2(1-h)$  if  $1 < h \leq 2$ .

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## 2. Definition of bifractional Brownian motion with parameter $K \in (1, 2)$

For any  $K \in (0, 2)$ , let  $X^K = (X_t^K, t \geq 0)$  be a Gaussian process defined by

$$X_t^K = \int_0^\infty (1 - e^{-rt}) r^{-\frac{1+K}{2}} dW_r, \quad t \geq 0 \quad (2.1)$$

where  $(W_t, t \geq 0)$  is a standard Brownian motion.

This process was introduced in [10] for  $K \in (0, 1)$  in order to obtain a decomposition of the bifractional Brownian motion with  $H \in (0, 1)$  and  $K \in (0, 1)$ . More precisely, they prove the following result:

**Theorem 2.1** (see [10]). *Let  $B^{H,K}$  a bifractional Brownian motion with parameters  $H \in (0, 1)$  and  $K \in (0, 1)$ ,  $B^{HK}$  be a fractional Brownian motion with Hurst parameter  $HK \in (0, 1)$  and  $W = \{W_t, t \geq 0\}$  a standard Brownian motion. Let  $X^K$  be the process given by (2.1). If we suppose that  $B^{H,K}$  and  $W$  are independents, then the processes  $\{Y_t = C_1 X_{t^{2H}}^K + B_t^{H,K}, t \geq 0\}$  and  $\{C_2 B_t^{HK}, t \geq 0\}$  have the same distribution, where  $C_1 = \sqrt{\frac{2^{-K}K}{\Gamma(1-K)}}$  and  $C_2 = 2^{\frac{1-K}{2}}$ .*

The process defined in (2.1) has good properties. The following result is proved in [10] for the case  $K \in (0, 1)$  and extended to the case  $K \in (1, 2)$  in [2] and [11]:

**Proposition 2.2** (see [2],[10] and [11]). *The process  $X^K = \{X_t^K, t \geq 0\}$  is Gaussian, centered, and its covariance function is:*

$$\text{Cov}(X_t^K, X_s^K) = \begin{cases} \frac{\Gamma(1-K)}{K} [t^K + s^K - (t+s)^K] & \text{if } K \in (0, 1), \\ \frac{\Gamma(2-K)}{K(K-1)} [(t+s)^K - t^K - s^K] & \text{if } K \in (1, 2). \end{cases} \quad (2.2)$$

Moreover,  $X^K$  has a version with trajectories which are infinitely differentiable on  $(0, \infty)$  and absolutely continuous on  $[0, \infty)$ .

Using the fact that when  $K \in (1, 2)$ , the covariance function of  $X^K$  is given by

$$\text{Cov}(X_t^K, X_s^K) = \frac{\Gamma(2-K)}{K(K-1)} ((t+s)^K - t^K - s^K),$$

and considering also the process

$$X_t^{H,K} = X_{t^{2H}}^K; \quad t \geq 0, \quad (2.3)$$

we can prove the following result:

**Theorem 2.3.** *Assume  $H \in (0, 1)$  and  $K \in (1, 2)$  with  $HK \in (0, 1)$ . Let  $B^{HK}$  be a fractional Brownian motion, and  $W = \{W_t, t \geq 0\}$  a standard Brownian motion. Let  $X^{K,H}$  the process defined in (2.3). If we suppose that  $B^{HK}$  and  $W$  are independents, then the processes*

$$B_t^{H,K} = aB_t^{HK} + bX_t^{H,K}, \quad (2.4)$$

where  $a = \sqrt{2^{1-K}}$  and  $b = \sqrt{\frac{K(K-1)}{2^K\Gamma(2-K)}}$  is a centered Gaussian process with covariance function

$$E\left(B_t^{H,K} B_s^{H,K}\right) = \frac{1}{2^K} \left( (t^{2H} + s^{2H})^K - |t-s|^{2HK} \right); \quad s, t \geq 0.$$

*Proof.* It is obvious that the process defined in (2.4) is a centered Gaussian process. On the other hand, its covariance functions is given by

$$\begin{aligned} E\left(B_t^{H,K} B_s^{H,K}\right) &= a^2 E\left(B_t^{HK} B_s^{HK}\right) + b^2 E\left(X_t^{H,K} X_s^{H,K}\right) \\ &= \frac{1}{2^K} \left(t^{2HK} + s^{2HK} - |t-s|^{2HK}\right) \\ &\quad + \frac{1}{2^K} \left(\left(t^{2H} + s^{2H}\right)^K - t^{2HK} - s^{2HK}\right) \\ &= \frac{1}{2^K} \left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right), \end{aligned}$$

which completes the proof.  $\square$

Thus, the bifractional Brownian motion  $B^{H,K}$  with parameters  $H \in (0,1)$  and  $K \in (1,2)$  such that  $HK \in (0,1)$  is well defined and it has a decomposition as a sum of a fBm  $B^{HK}$  and an absolutely continuous process  $X^{H,K}$ .

*Remark 2.4.* Assume that  $2HK = 1$ . Russo and Tudor [12] proved that if  $K$  belong to  $(0,1)$ , the process  $B^{H,K}$  is not a semimartingale. But in the case when  $1 < K < 2$ ,  $B^{H,K}$  is a semimartingale because we have a decomposition of this process as a sum of a Brownian motion  $B^{\frac{1}{2}}$  and a finite variation process  $X^{H,K}$ .

The following decomposition is exploited to prove the quasi-helix property (in the sense of J.P. Kahane) of  $B^{H,K}$ . This result is satisfied for all  $K \in (0,2)$ .

**Proposition 2.5.** *Let  $H \in (0,1)$  and  $K \in (0,2)$  such that  $HK \in (0,1)$ . Let  $(\xi_t^{K/2}, t \geq 0)$  be a sub-fractional Brownian motion with parameter  $K/2 \in (0,1)$ , independent to  $B^{H,K}$  and suppose that  $(B_t^{K/2}, t \geq 0)$  and  $(B_t^{HK}, t \geq 0)$  are two independent fractional Brownian motions with Hurst parameter  $K/2 \in (0,1)$  and  $HK \in (0,1)$ , respectively. We set  $\xi_t^{K,H} = \xi_{t^{2H}}^{K/2}$  and  $\tilde{B}_t^{H,K} = B_{t^{2H}}^{K/2}$ ,  $t \geq 0$ . Then, it holds that*

$$B^{H,K} + \sqrt{2^{1-K}} \xi^{K,H} \stackrel{(d)}{=} \sqrt{2^{1-K}} \left(\tilde{B}^{H,K} + B^{HK}\right) \quad (2.5)$$

where  $\stackrel{(d)}{=}$  denotes that both processes have the same distribution.

*Proof.* The result follows easily from the independence and the fact that their corresponding covariance functions satisfy the following equality for all  $s, t \geq 0$

$$\begin{aligned} R^{H,K}(t, s) &= \frac{1}{2^K} \left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right) \\ &= 2^{1-K} \left[-Cov(\xi_t^{K,H}, \xi_s^{K,H}) + Cov(\tilde{B}_t^{H,K}, \tilde{B}_s^{H,K})\right. \\ &\quad \left.+ Cov(B_t^{HK}, B_s^{HK})\right]. \end{aligned}$$

$\square$

**Proposition 2.6.** *Let  $H \in (0,1)$  and  $K \in (1,2)$  such that  $HK \in (0,1)$ . Then for any  $t, s \geq 0$ , if  $0 < H \leq 1/2$*

$$2^{1-K} |t-s|^{2HK} \leq E\left(B_t^{H,K} - B_s^{H,K}\right)^2 \leq |t-s|^{2HK},$$

and if  $1/2 < H < 1$

$$2^{1-K}|t-s|^{2HK} \leq E\left(B_t^{H,K} - B_s^{H,K}\right)^2 \leq 2^{2-K}|t-s|^{2HK}.$$

*Proof.* Using the proposition 2.5, we obtain

$$\begin{aligned} & E\left(B_t^{H,K} - B_s^{H,K}\right)^2 \\ &= 2^{1-K} \left( -E\left(\xi_{t^{2H}}^{\frac{K}{2}} - \xi_{s^{2H}}^{\frac{K}{2}}\right)^2 + E\left(B_{t^{2H}}^{\frac{K}{2}} - B_{s^{2H}}^{\frac{K}{2}}\right)^2 + E\left(B_t^{HK} - B_s^{HK}\right)^2 \right) \\ &= 2^{1-K} \left( -E\left(\xi_{t^{2H}}^{\frac{K}{2}} - \xi_{s^{2H}}^{\frac{K}{2}}\right)^2 + |t^{2H} - s^{2H}|^K + |t-s|^{2HK} \right). \end{aligned}$$

On the other hand, from [3] we have

$$(2 - 2^{K-1})|t^{2H} - s^{2H}|^K \leq E\left(\xi_{t^{2H}}^{\frac{K}{2}} - \xi_{s^{2H}}^{\frac{K}{2}}\right)^2 \leq |t^{2H} - s^{2H}|^K.$$

Thus

$$\begin{aligned} 2^{1-K}|t-s|^{2HK} &\leq E\left(B_t^{H,K} - B_s^{H,K}\right)^2 \\ &\leq 2^{1-K} \left( |t-s|^{2HK} + (2^{K-1} - 1)|t^{2H} - s^{2H}|^K \right). \end{aligned}$$

Then we deduce that for every  $H \in (0, 1)$ ,  $K \in (1, 2)$  with  $HK \in (0, 1)$

$$2^{1-K}|t-s|^{2HK} \leq E\left(B_t^{H,K} - B_s^{H,K}\right)^2$$

and the other hand for every  $H \in (0, \frac{1}{2}]$ ,  $K \in (1, 2)$  we have

$$\begin{aligned} E\left(B_t^{H,K} - B_s^{H,K}\right)^2 &\leq 2^{1-K} \left( |t-s|^{2HK} + (2^{K-1} - 1)|t^{2H} - s^{2H}|^K \right) \\ &\leq |t-s|^{2HK}. \end{aligned}$$

The last inequality is satisfied from the fact that  $|t^{2H} - s^{2H}| \leq |t-s|^{2H}$  for  $H \in (0, \frac{1}{2}]$ .

To complete the proof, it remains to show that for every  $H \in (\frac{1}{2}, 1)$ ,  $K \in (1, 2)$  with  $HK \in (0, 1)$  (observe that in this situation we have  $HK \in (\frac{1}{2}, 1)$ )

$$E\left(B_t^{H,K} - B_s^{H,K}\right)^2 \leq 2^{2-K}|t-s|^{2HK}.$$

Notice that,

$$\begin{aligned} E\left(B_t^{H,K} - B_s^{H,K}\right)^2 &= \frac{1}{2^K} \left[ (2t^{2H})^K + (2s^{2H})^K \right. \\ &\quad \left. - 2((t^{2H} + s^{2H})^K - |t-s|^{2HK}) \right] \\ &= \frac{2}{2^K}|t-s|^{2HK} + \left( t^{2HK} + s^{2HK} - \frac{2}{2^K}(t^{2H} + s^{2H})^K \right). \end{aligned}$$

Hence it is enough to prove that

$$t^{2HK} + s^{2HK} - \frac{2}{2^K}(t^{2H} + s^{2H})^K \leq 2^{1-K}|t-s|^{2HK},$$

or equivalently

$$t^{2HK} + s^{2HK} \leq 2^{1-K} ((t^{2H} + s^{2H})^K + |t - s|^{2HK}).$$

From now on we will assume, without loss of generality, that  $s \leq t$ . Dividing by  $t^{2HK}$  we obtain that we have to prove that

$$1 + \left(\frac{s}{t}\right)^{2HK} \leq 2^{1-K} \left( \left(1 + \left(\frac{s}{t}\right)^{2H}\right)^K + \left(1 - \frac{s}{t}\right)^{2HK} \right).$$

Equivalently we have to prove that, for any  $u \in (0, 1]$  the function

$$f(u) := 2^{1-K} [(1 + u^{2H})^K + (1 - u)^{2HK}] - u^{2HK} - 1$$

is positive.

Observe that  $f(1) = 0$ , so, it is enough to see that the derivative of this function is negative for  $u \in (0, 1]$ . But,

$$f'(u) = 2HK2^{1-K}u^{2HK-1} \left[ \left(1 + \frac{1}{u^{2H}}\right)^{K-1} - \left(\frac{1}{u} - 1\right)^{2HK-1} - 2^{K-1} \right].$$

To prove that  $f'(u) \leq 0$  for  $u \in (0, 1]$  it is enough to see that the function

$$h(u) := \left(1 + \frac{1}{u^{2H}}\right)^{K-1} - \left(\frac{1}{u} - 1\right)^{2HK-1} - 2^{K-1},$$

is negative for  $u \in (0, 1]$ . But, since  $h(1) = 0$ , it is enough to prove that its derivative  $h'(u) \geq 0$  for  $u \in (0, 1]$ . But,

$$h'(u) = \frac{1}{u^{2HK}} (-2(K-1)H(u^{2H} + 1)^{K-2}u^{2H-1} + (1-u)^{2HK-2}(2HK-1)).$$

Observe that  $u^{2H-1} \leq 1$  because  $H \in (\frac{1}{2}, 1)$ ,  $(u^{2H} + 1)^{K-2} \leq 1$  and

$$(1-u)^{2HK-2} \geq 1.$$

So,

$$h'(u) \geq \frac{1}{u^{2HK}} (-2(K-1)H + 2HK - 1) = \frac{1}{u^{2HK}} (2H - 1) \geq 0,$$

because  $H \geq \frac{1}{2}$ . The prove is now complete.  $\square$

**Proposition 2.7.** *Suppose that  $H \in (0, 1)$ ,  $K \in (1, 2)$  such that  $HK \in (0, 1)$ . The bifBm  $B^{H,K}$  has the following properties*

i)  $B^{H,K}$  is a self-similar process with index  $HK$ , i.e.

$$\left(B_{at}^{H,K}, t \geq 0\right) \stackrel{d}{=} \left(a^{HK} B_t^{H,K}, t \geq 0\right), \quad \text{for each } a > 0.$$

ii)  $B^{H,K}$  has the same long-range property of the fBm  $B^{HK}$ , i.e.  $B^{H,K}$  has the short-memory for  $HK < \frac{1}{2}$  and it has long-memory for  $HK > \frac{1}{2}$ .

iii)  $B^{H,K}$  has a  $\frac{1}{HK}$ -variation equals to  $2^{\frac{1-K}{HK}} \lambda t$  with  $\lambda = E(|N|^{\frac{1}{HK}})$  and  $N$  being a standard normal random variable, i.e.

$$\sum_{j=1}^n \left(B_{t_j^n}^{H,K} - B_{t_{j-1}^n}^{H,K}\right)^{\frac{1}{HK}} \xrightarrow[n \rightarrow \infty]{} 2^{\frac{1-K}{HK}} \lambda t \quad \text{in } L^1(\Omega).$$

where  $0 = t_0^n < \dots < t_n^n = t$  denotes a partition of  $[0, t]$ .

iv)  $B^{H,K}$  is not a semimartingale if  $2HK \neq 1$ .

The proof of the proposition 2.7 is straightforward from [12] and [6].

### 3. Space of integrable functions with respect to bifractional Brownian motion

Let us consider  $\mathcal{E}$  the set of simple functions on  $[0, T]$ . Generally, if  $U := (U_t, t \in [0, T])$  is a continuous, centered Gaussian process, we denote by  $\mathcal{H}_U$  the Hilbert space defined as the closure of  $\mathcal{E}$  with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = E(U_t U_s).$$

In the case of the standard Brownian motion  $W$ , the space  $\mathcal{H}_W$  is  $L^2([0, T])$ . On the other hand, for the fractional Brownian motion  $B^H$ , the space  $\mathcal{H}_{B^H}$  is the set of restrictions to the space of test functions  $\mathcal{D}((0, T))$  of the distributions of  $W^{\frac{1}{2}-H, 2}(\mathbb{R})$  with support contained in  $[0, T]$  (see [8]). In the case  $H \in (0, \frac{1}{2})$  all the elements of the domain are functions, and the space  $\mathcal{H}_{B^H}$  coincides with the fractional Sobolev space  $I_{0+}^{\frac{1}{2}-H}(L^2([0, T]))$  (see for instance [5]), but in the case  $H \in (\frac{1}{2}, 1)$  this space contains distributions which are not given by any function.

As a direct consequence of Theorem 2.3 we have the following relation between  $\mathcal{H}_{B^H}$ ,  $\mathcal{H}_{B^{H,K}}$  and  $\mathcal{H}_{X^{H,K}}$ , where  $B^{H,K}$  is the bifractional Brownian motion and  $X^{H,K}$  is the process defined in (2.3).

**Proposition 3.1.** *Let  $H \in (0, 1)$  and  $K \in (1, 2)$  with  $HK \in (0, 1)$ . Then it holds that*

$$\mathcal{H}_{X^{H,K}} \cap \mathcal{H}_{B^{H,K}} = \mathcal{H}_{B^{H,K}}$$

If we consider the processes appearing in Proposition 2.5 we have also the following result:

**Proposition 3.2.** *Let  $H \in (0, 1)$ . For every  $K \in (0, 2)$  with  $HK \in (0, 1)$  the following equality holds*

$$\mathcal{H}_{\xi^{H,K}} \cap \mathcal{H}_{B^{H,K}} = \mathcal{H}_{\bar{B}^{H,K}} \cap \mathcal{H}_{B^{H,K}}.$$

*Proof.* Both propositions are a direct consequence of the two decompositions into the sum of two independent processes proved in Theorem 2.3 and Proposition 2.5.  $\square$

*Remark 3.3.* For the case  $K \in (0, 1)$  we have the following equality (see [10])

$$\mathcal{H}_{B^{H,K}} = \mathcal{H}_{X^{H,K}} \cap \mathcal{H}_{B^{H,K}}.$$

### 4. Weak convergence towards the bifractional Brownian motion

Another direct consequence of the decomposition for the bifractional Brownian motion with  $H \in (0, 1)$ ,  $K \in (1, 2)$  and  $HK \in (0, 1)$  is the following result of convergence in law in the space  $\mathcal{C}([0, T])$ .

Recall that the fractional Brownian motion of Hurst parameter  $H \in (0, 1)$  admits an integral representation of the form (see for instance [1])

$$B_t^H = \int_0^t K^H(t, s) dW_s,$$

where  $W$  is a standard Brownian motion and the kernel  $K^H$  is defined on the set  $\{0 < s < t\}$  and given by

$$K^H(t, s) = d_H(t-s)^{H-\frac{1}{2}} + d_H\left(\frac{1}{2} - H\right) \int_s^t (u-s)^{H-\frac{3}{2}} \left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right) du, \quad (4.1)$$

with  $d_H$  the following normalizing constant

$$d_H = \left( \frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2})\Gamma(2 - 2H)} \right)^{\frac{1}{2}}.$$

**Theorem 4.1.** *Let  $H \in (0, 1)$  and  $K \in (1, 2)$  with  $HK \in (0, 1)$ . Consider  $\theta \in (0, \pi) \cup (\pi, 2\pi)$  such that if  $HK \in (0, \frac{1}{4}]$  then  $\theta$  satisfies that  $\cos((2i+1)\theta) \neq 1$  for all  $i \in \mathbb{N}$  such that  $i \leq \frac{1}{4} \lceil \frac{1}{HK} \rceil$ . Set  $a = \sqrt{2^{1-K}}$  and  $b = \sqrt{\frac{K(K-1)}{2^K\Gamma(2-K)}}$ . Define the processes,*

$$\begin{aligned} B_\epsilon^{HK} &= \left\{ \frac{2}{\epsilon} \int_0^T K^{HK}(t, s) \sin\left(\theta N_{\frac{2s}{\epsilon^2}}\right) ds, \quad t \in [0, T] \right\}, \\ X_\epsilon^{H,K} &= \left\{ \frac{2}{\epsilon} \int_0^\infty (1 - e^{-st^{2H}}) s^{-\frac{1+K}{2}} \cos\left(\theta N_{\frac{2s}{\epsilon^2}}\right) ds, \quad t \in [0, T] \right\}, \end{aligned}$$

where  $K^{HK}(t, s)$  is the kernel defined in (4.1). Then,

$$\{Y_\epsilon^H(t) = aB_\epsilon^{HK}(t) + bX_\epsilon^{H,K}(t), t \in [0, T]\}$$

weakly converges in  $\mathcal{C}([0, T])$  to a bifractional Brownian motion.

*Proof.* Applying Theorems 3.2 and 3.5 of [2] we know that, respectively, the processes  $B_\epsilon^{HK}$  and  $X_\epsilon^{H,K}$  converge in law in  $\mathcal{C}([0, T])$  towards a fBm  $B^{HK}$  and to the process  $X^{H,K}$ . Moreover, applying Theorem 2.1 of [2], we know that the limit laws are independent. Hence, we are under the hypothesis of the decomposition obtained in Theorem 2.3, which proves the stated result.  $\square$

*Remark 4.2.* Obviously we can also obtain the same result interchanging the roles of the sinus and the cosinus functions in the definition of the approximating processes.

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