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Bifractional Brownian motion with parameter $K \in (1, 2)$

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Abstract

In this paper we introduce and study a self-similar Gaussian process that is the bifractional Brownian motion $B^{H,K}$ with parameters $H \in (0, 1)$ and $K \in (1, 2)$ such that $HK \in (0, 1)$. A remarkable difference between the case $K \in (0, 1)$ and our situation is that this process is a semimartingale when $2HK = 1$.

Key words and phrases: Bifractional Brownian motion, self-similar processes, long-range dependence.

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1 Introduction

Houdré and Villa in [7] gave the first introduction to the bifractional Brownian motion (bifBm) $B^{H,K} = (B_t^{H,K}; t \geq 0)$ with parameters $H \in (0, 1)$ and $K \in (0, 1]$ which is defined as a centered Gaussian process, with covariance function

$$R^{H,K}(t, s) = E \left(B_t^{H,K} B_s^{H,K} \right) = \frac{1}{2K} \left((t^{2H} + s^{2H})^K - |t - s|^{2HK} \right); \quad s, t \geq 0.$$

The case $K = 1$ corresponds to the fractional Brownian motion (fBm) with Hurst parameter H . Some properties of the bifractional Brownian motion have been studied by Russo and

Tudor in [12]. In fact, in [12] it is shown that the bifractional Brownian motion behaves as a fractional Brownian motion with Hurst parameter HK . A stochastic calculus with respect to this process has been recently developed by Kruk, Russo and Tudor [9] and Es-Sebaiy and Tudor [6].

In this paper we prove that, with $H \in (0, 1)$ and $HK \in (0, 1)$, the process $B^{H,K}$ can be extended for $1 < K < 2$. The case $H = \frac{1}{2}$ and $1 < K < 2$ plays a role to give an extension of sub-fractional Brownian motion (subfBm) (see [4]). The subfBm $(\xi_t^h, t \geq 0)$ with parameter $0 < h \leq 2$ is a centered Gaussian process with covariance:

$$E(\xi_t^h \xi_s^h) = C_h \left(t^{2h} + s^{2h} - \frac{1}{2} ((t+s)^{2h} + |t-s|^{2h}) \right); \quad s, t \geq 0$$

where $C_h = 1$ if $0 < h < 1$ and $C_h = 2(1-h)$ if $1 < h \leq 2$.

2 Definition of bifractional Brownian motion with parameter $K \in (1, 2)$

For any $K \in (0, 2)$, let $X^K = (X_t^K, t \geq 0)$ be a Gaussian process defined by

$$X_t^K = \int_0^\infty (1 - e^{-rt}) r^{-\frac{1+K}{2}} dW_r, \quad t \geq 0 \quad (2.1)$$

where $(W_t, t \geq 0)$ is a standard Brownian motion.

This process was introduced in [10] for $K \in (0, 1)$ in order to obtain a decomposition of the bifractional Brownian motion with $H \in (0, 1)$ and $K \in (0, 1)$. More precisely, they prove the following result:

Theorem 1 (see [10]) *Let $B^{H,K}$ a bifractional Brownian motion with parameters $H \in (0, 1)$ and $K \in (0, 1)$, B^{HK} be a fractional Brownian motion with Hurst parameter $HK \in (0, 1)$ and $W = \{W_t, t \geq 0\}$ a standard Brownian motion. Let X^K be the process given by (2.1). If we suppose that $B^{H,K}$ and W are independents, then the processes $\{Y_t = C_1 X_{t^{2H}}^K + B_t^{H,K}, t \geq 0\}$ and $\{C_2 B_t^{HK}, t \geq 0\}$ have the same distribution, where $C_1 = \sqrt{\frac{2-K}{\Gamma(1-K)}}$ and $C_2 = 2^{\frac{1-K}{2}}$.*

The process defined in (2.1) has good properties. The following result is proved in [10] for the case $K \in (0, 1)$ and extended to the case $K \in (1, 2)$ in [2] and [11]:

Proposition 1 (see [2],[10] and [11]) *The process $X^K = \{X_t^K, t \geq 0\}$ is Gaussian, centered, and its covariance function is:*

$$\text{Cov}(X_t^K, X_s^K) = \begin{cases} \frac{\Gamma(1-K)}{K} [t^K + s^K - (t+s)^K] & \text{if } K \in (0, 1), \\ \frac{\Gamma(2-K)}{K(K-1)} [(t+s)^K - t^K - s^K] & \text{if } K \in (1, 2). \end{cases} \quad (2.2)$$

Moreover, X^K has a version with trajectories which are infinitely differentiable on $(0, \infty)$ and absolutely continuous on $[0, \infty)$.

Using the fact that when $K \in (1, 2)$, the covariance function of X^K is given by

$$\text{Cov}(X_t^K, X_s^K) = \frac{\Gamma(2-K)}{K(K-1)} ((t+s)^K - t^K - s^K),$$

and considering also the process

$$X_t^{H,K} = X_{t^{2H}}^K; \quad t \geq 0, \quad (2.3)$$

we can prove the following result:

Theorem 2 *Assume $H \in (0, 1)$ and $K \in (1, 2)$ with $HK \in (0, 1)$. Let B^{HK} be a fractional Brownian motion, and $W = \{W_t, t \geq 0\}$ a standard Brownian motion. Let $X^{K,H}$ the process defined in (2.3). If we suppose that B^{HK} and W are independents, then the processes*

$$B_t^{H,K} = aB_t^{HK} + bX_t^{H,K}, \quad (2.4)$$

where $a = \sqrt{2^{1-K}}$ and $b = \sqrt{\frac{K(K-1)}{2^K \Gamma(2-K)}}$ is a centered Gaussian process with covariance function

$$E\left(B_t^{H,K} B_s^{H,K}\right) = \frac{1}{2^K} \left((t^{2H} + s^{2H})^K - |t-s|^{2HK} \right); \quad s, t \geq 0.$$

Proof: It is obvious that the process defined in (2.4) is a centered Gaussian process. On the other hand, its covariance functions is given by

$$\begin{aligned} E\left(B_t^{H,K} B_s^{H,K}\right) &= a^2 E\left(B_t^{HK} B_s^{HK}\right) + b^2 E\left(X_t^{H,K} X_s^{H,K}\right) \\ &= \frac{1}{2^K} (t^{2HK} + s^{2HK} - |t-s|^{2HK}) + \frac{1}{2^K} \left((t^{2H} + s^{2H})^K - t^{2HK} - s^{2HK} \right) \\ &= \frac{1}{2^K} \left((t^{2H} + s^{2H})^K - |t-s|^{2HK} \right), \end{aligned}$$

which completes the proof. \square

Thus, the bifractional Brownian motion $B^{H,K}$ with parameters $H \in (0, 1)$ and $K \in (1, 2)$ such that $HK \in (0, 1)$ is well defined and it has a decomposition as a sum of a fBm B^{HK} and an absolutely continuous process $X^{H,K}$.

Remark 1 *Assume that $2HK = 1$. Russo and Tudor [12] proved that if $0 < K < 1$, the process $B^{H,K}$ is not a semimartingale. But in the case when $1 < K < 2$, $B^{H,K}$ is a semimartingale because we have a decomposition of this process as a sum of a Brownian motion $B^{\frac{1}{2}}$ and a finite variation process $X^{H,K}$.*

The following decomposition is exploited to prove the quasi-helix property (in the sense of J.P. Kahane) of $B^{H,K}$. This result is satisfied for all $K \in (0, 2)$.

Proposition 2 Let $H \in (0, 1)$ and $K \in (0, 2)$ such that $HK \in (0, 1)$. Let $(\xi_t^{K/2}, t \geq 0)$ be a sub-fractional Brownian motion with parameter $K/2 \in (0, 1)$, independent to $B^{H,K}$ and suppose that $(B_t^{K/2}, t \geq 0)$ and $(B_t^{HK}, t \geq 0)$ are two independent fractional Brownian motions with Hurst parameter $K/2 \in (0, 1)$ and $HK \in (0, 1)$, respectively. We set $\xi_t^{K,H} = \xi_{t^{2H}}^{K/2}$ and $\tilde{B}_t^{H,K} = B_{t^{2H}}^{K/2}$, $t \geq 0$. Then, it holds that

$$B^{H,K} + \sqrt{2^{1-K}} \xi^{K,H} \stackrel{(d)}{=} \sqrt{2^{1-K}} \left(\tilde{B}^{H,K} + B^{HK} \right) \quad (2.5)$$

where $\stackrel{(d)}{=}$ denotes that both processes have the same distribution.

Proof: The result follows easily from the independence and the fact that their corresponding covariance functions satisfy the following equality for all $s, t \geq 0$

$$\begin{aligned} R^{H,K}(t, s) &= \frac{1}{2^K} \left((t^{2H} + s^{2H})^K - |t - s|^{2HK} \right) \\ &= 2^{1-K} \left(-\text{Cov}(\xi_t^{K,H}, \xi_s^{K,H}) + \text{Cov}(\tilde{B}_t^{H,K}, \tilde{B}_s^{H,K}) + \text{Cov}(B_t^{HK}, B_s^{HK}) \right). \end{aligned}$$

□

Proposition 3 Let $H \in (0, 1)$ and $K \in (1, 2)$ such that $HK \in (0, 1)$. Then for any $t, s \geq 0$, if $0 < H \leq 1/2$

$$2^{1-K} |t - s|^{2HK} \leq E \left(B_t^{H,K} - B_s^{H,K} \right)^2 \leq |t - s|^{2HK},$$

and if $1/2 < H < 1$

$$2^{1-K} |t - s|^{2HK} \leq E \left(B_t^{H,K} - B_s^{H,K} \right)^2 \leq 2^{2-K} |t - s|^{2HK}.$$

Proof: Using the proposition 2, we obtain

$$\begin{aligned} &E \left(B_t^{H,K} - B_s^{H,K} \right)^2 \\ &= 2^{1-K} \left(-E \left(\xi_{t^{2H}}^{K/2} - \xi_{s^{2H}}^{K/2} \right)^2 + E \left(B_{t^{2H}}^{K/2} - B_{s^{2H}}^{K/2} \right)^2 + E \left(B_t^{HK} - B_s^{HK} \right)^2 \right) \\ &= 2^{1-K} \left(-E \left(\xi_{t^{2H}}^{K/2} - \xi_{s^{2H}}^{K/2} \right)^2 + |t^{2H} - s^{2H}|^K + |t - s|^{2HK} \right). \end{aligned}$$

On the other hand, from [3] we have

$$(2 - 2^{K-1}) |t^{2H} - s^{2H}|^K \leq E \left(\xi_{t^{2H}}^{K/2} - \xi_{s^{2H}}^{K/2} \right)^2 \leq |t^{2H} - s^{2H}|^K.$$

Thus

$$2^{1-K} |t-s|^{2HK} \leq E \left(B_t^{H,K} - B_s^{H,K} \right)^2 \leq 2^{1-K} \left(|t-s|^{2HK} + (2^{K-1} - 1) |t^{2H} - s^{2H}|^K \right).$$

Then we deduce that for every $H \in (0, 1)$, $K \in (1, 2)$ with $HK \in (0, 1)$

$$2^{1-K} |t-s|^{2HK} \leq E \left(B_t^{H,K} - B_s^{H,K} \right)^2$$

and the other hand for every $H \in (0, \frac{1}{2}]$, $K \in (1, 2)$ we have

$$\begin{aligned} E \left(B_t^{H,K} - B_s^{H,K} \right)^2 &\leq 2^{1-K} \left(|t-s|^{2HK} + (2^{K-1} - 1) |t^{2H} - s^{2H}|^K \right) \\ &\leq |t-s|^{2HK}. \end{aligned}$$

The last inequality is satisfied from the fact that $|t^{2H} - s^{2H}| \leq |t-s|^{2H}$ for $H \in (0, \frac{1}{2}]$. To complete the proof, it remains to show that for every $H \in (\frac{1}{2}, 1)$, $K \in (1, 2)$ with $HK \in (0, 1)$ (observe that in this situation we have $HK \in (\frac{1}{2}, 1)$)

$$E \left(B_t^{H,K} - B_s^{H,K} \right)^2 \leq 2^{2-K} |t-s|^{2HK}.$$

Notice that,

$$\begin{aligned} E \left(B_t^{H,K} - B_s^{H,K} \right)^2 &= \frac{1}{2^K} \left((2t^{2H})^K + (2s^{2H})^K - 2 \left((t^{2H} + s^{2H})^K - |t-s|^{2HK} \right) \right) \\ &= \frac{2}{2^K} |t-s|^{2HK} + \left(t^{2HK} + s^{2HK} - \frac{2}{2^K} (t^{2H} + s^{2H})^K \right). \end{aligned}$$

Hence it is enough to prove that

$$t^{2HK} + s^{2HK} - \frac{2}{2^K} (t^{2H} + s^{2H})^K \leq 2^{1-K} |t-s|^{2HK},$$

or equivalently

$$t^{2HK} + s^{2HK} \leq 2^{1-K} \left((t^{2H} + s^{2H})^K + |t-s|^{2HK} \right).$$

From now on we will assume, without loss of generality, that $s \leq t$. Dividing by t^{2HK} we obtain that we have to prove that

$$1 + \left(\frac{s}{t} \right)^{2HK} \leq 2^{1-K} \left(\left(1 + \left(\frac{s}{t} \right)^{2H} \right)^K + \left(1 - \frac{s}{t} \right)^{2HK} \right).$$

Equivalently we have to prove that, for any $u \in (0, 1]$ the function

$$f(u) := 2^{1-K} \left[(1 + u^{2H})^K + (1 - u)^{2HK} \right] - u^{2HK} - 1$$

is positive.

Observe that $f(1) = 0$, so, it is enough to see that the derivative of this function is negative for $u \in (0, 1]$. But,

$$f'(u) = 2HK2^{1-K}u^{2HK-1} \left[\left(1 + \frac{1}{u^{2H}}\right)^{K-1} - \left(\frac{1}{u} - 1\right)^{2HK-1} - 2^{K-1} \right].$$

To prove that $f'(u) \leq 0$ for $u \in (0, 1]$ it is enough to see that the function

$$h(u) := \left(1 + \frac{1}{u^{2H}}\right)^{K-1} - \left(\frac{1}{u} - 1\right)^{2HK-1} - 2^{K-1},$$

is negative for $u \in (0, 1]$. But, since $h(1) = 0$, it is enough to prove that its derivative $h'(u) \geq 0$ for $u \in (0, 1]$. But,

$$h'(u) = \frac{1}{u^{2HK}} (-2(K-1)H(u^{2H} + 1)^{K-2}u^{2H-1} + (1-u)^{2HK-2}(2HK-1)).$$

Observe that $u^{2H-1} \leq 1$ because $H \in (\frac{1}{2}, 1)$, $(u^{2H} + 1)^{K-2} \leq 1$ and $(1-u)^{2HK-2} \geq 1$. So,

$$h'(u) \geq \frac{1}{u^{2HK}} (-2(K-1)H + 2HK - 1) = \frac{1}{u^{2HK}} (2H - 1) \geq 0,$$

because $H \geq \frac{1}{2}$. The prove is now complete. \square

Proposition 4 *Suppose that $H \in (0, 1)$, $K \in (1, 2)$ such that $HK \in (0, 1)$. The bifBm $B^{H,K}$ has the following properties*

i) $B^{H,K}$ is a self-similar process with index HK , i.e.

$$\left(B_{at}^{H,K}, t \geq 0\right) \stackrel{d}{=} \left(a^{HK} B_t^{H,K}, t \geq 0\right), \quad \text{for each } a > 0.$$

ii) $B^{H,K}$ has the same long-range property of the fBm B^{HK} , i.e. $B^{H,K}$ has the short-memory for $HK < \frac{1}{2}$ and it has long-memory for $HK > \frac{1}{2}$.

iii) $B^{H,K}$ has a $\frac{1}{HK}$ -variation equals to $2^{\frac{1-K}{HK}} \lambda t$ with $\lambda = E(|N|^{\frac{1}{HK}})$ and N being a standard normal random variable, i.e.

$$\sum_{j=1}^n \left(B_{t_j^n}^{H,K} - B_{t_{j-1}^n}^{H,K}\right)^{\frac{1}{HK}} \xrightarrow[n \rightarrow \infty]{} 2^{\frac{1-K}{HK}} \lambda t \quad \text{in } L^1(\Omega).$$

where $0 = t_0^n < \dots < t_n^n = t$ denotes a partition of $[0, t]$.

iv) $B^{H,K}$ is not a semimartingale if $2HK \neq 1$.

The proof of the proposition 4 is straightforward from [12] and [6].

3 Space of integrable functions with respect to bifractional Brownian motion

Let us consider \mathcal{E} the set of simple functions on $[0, T]$. Generally, if $U := (U_t, t \in [0, T])$ is a continuous, centered Gaussian process, we denote by \mathcal{H}_U the Hilbert space defined as the closure of \mathcal{E} with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = E(U_t U_s).$$

In the case of the standard Brownian motion W , the space \mathcal{H}_W is $L^2([0, T])$. On the other hand, for the fractional Brownian motion B^H , the space \mathcal{H}_{B^H} is the set of restrictions to the space of test functions $\mathcal{D}((0, T))$ of the distributions of $W^{\frac{1}{2}-H, 2}(\mathbb{R})$ with support contained in $[0, T]$ (see [8]). In the case $H \in (0, \frac{1}{2})$ all the elements of the domain are functions, and the space \mathcal{H}_{B^H} coincides with the fractional Sobolev space $I_{0+}^{\frac{1}{2}-H}(L^2([0, T]))$ (see for instance [5]), but in the case $H \in (\frac{1}{2}, 1)$ this space contains distributions which are not given by any function.

As a direct consequence of Theorem 2 we have the following relation between \mathcal{H}_{B^H} , $\mathcal{H}_{B^{H,K}}$ and $\mathcal{H}_{X^{H,K}}$, where $B^{H,K}$ is the bifractional Brownian motion and $X^{H,K}$ is the process defined in (2.3).

Proposition 5 *Let $H \in (0, 1)$ and $K \in (1, 2)$ with $HK \in (0, 1)$. Then it holds that*

$$\mathcal{H}_{X^{H,K}} \cap \mathcal{H}_{B^{HK}} = \mathcal{H}_{B^{H,K}}$$

If we consider the processes appearing in Proposition 2 we have also the following result:

Proposition 6 *Let $H \in (0, 1)$. For every $K \in (0, 2)$ with $HK \in (0, 1)$ the following equality holds*

$$\mathcal{H}_{\xi^{H,K}} \cap \mathcal{H}_{B^{H,K}} = \mathcal{H}_{\tilde{B}^{H,K}} \cap \mathcal{H}_{B^{HK}}.$$

Proof: Both propositions are a direct consequence of the two decompositions into the sum of two independent processes proved in Theorem 2 and Proposition 2. \square

Remark 2 *For the case $K \in (0, 1)$ we have the following equality (see [10])*

$$\mathcal{H}_{B^{HK}} = \mathcal{H}_{X^{H,K}} \cap \mathcal{H}_{B^{H,K}}.$$

4 Weak convergence towards the bifractional Brownian motion

Another direct consequence of the decomposition for the bifractional Brownian motion with $H \in (0, 1)$, $K \in (1, 2)$ and $HK \in (0, 1)$ is the following result of convergence in law in the space $\mathcal{C}([0, T])$.

Recall that the fractional Brownian motion of Hurst parameter $H \in (0, 1)$ admits an integral representation of the form (see for instance [1])

$$B_t^H = \int_0^t K^H(t, s) dW_s,$$

where W is a standard Brownian motion and the kernel K^H is defined on the set $\{0 < s < t\}$ and given by

$$K^H(t, s) = d_H(t-s)^{H-\frac{1}{2}} + d_H\left(\frac{1}{2}-H\right) \int_s^t (u-s)^{H-\frac{3}{2}} \left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right) du, \quad (4.6)$$

with d_H the following normalizing constant

$$d_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)} \right)^{\frac{1}{2}}.$$

Theorem 3 *Let $H \in (0, 1)$ and $K \in (1, 2)$ with $HK \in (0, 1)$. Consider $\theta \in (0, \pi) \cup (\pi, 2\pi)$ such that if $HK \in (0, \frac{1}{4}]$ then θ satisfies that $\cos((2i+1)\theta) \neq 1$ for all $i \in \mathbb{N}$ such that $i \leq \frac{1}{4} \lceil \frac{1}{H} \rceil$. Set $a = \sqrt{2^{1-K}}$ and $b = \sqrt{\frac{K(K-1)}{2^K\Gamma(2-K)}}$. Define the processes,*

$$\begin{aligned} B_\epsilon^{HK} &= \left\{ \frac{2}{\epsilon} \int_0^T K^{HK}(t, s) \sin\left(\theta N_{\frac{2s}{\epsilon^2}}\right) ds, \quad t \in [0, T] \right\}, \\ X_\epsilon^{H,K} &= \left\{ \frac{2}{\epsilon} \int_0^\infty (1 - e^{-st^{2H}}) s^{-\frac{1+K}{2}} \cos\left(\theta N_{\frac{2s}{\epsilon^2}}\right) ds, \quad t \in [0, T] \right\}, \end{aligned}$$

where $K^{HK}(t, s)$ is the kernel defined in (4.6). Then, $\{Y_\epsilon^H(t) = aB_\epsilon^{HK}(t) + bX_\epsilon^{H,K}(t), t \in [0, T]\}$ weakly converges in $\mathcal{C}([0, T])$ to a bifractional Brownian motion.

Proof: Applying Theorems 3.2 and 3.5 of [2] we know that, respectively, the processes B_ϵ^{HK} and $X_\epsilon^{H,K}$ converge in law in $\mathcal{C}([0, T])$ towards a fBm B^{HK} and to the process $X^{H,K}$. Moreover, applying Theorem 2.1 of [2], we know that the limit laws are independent. Hence, we are under the hypothesis of the decomposition obtained in Theorem 2, which proves the stated result.

Remark 3 *Obviously we can also obtain the same result interchanging the roles of the sinus and the cosinus functions in the definition of the approximating processes.*

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