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AN EXTENSION OF BIFRACTIONAL BROWNIAN MOTION

XAVIER BARDINA* AND KHALIFA ES-SEBAIY

Abstract. In this paper we introduce and study a self-similar Gaussian process that is the bifractional Brownian motion $B_{H,K}$ with parameters $H \in (0,1)$ and $K \in (1,2)$ such that $HK \in (0,1)$. A remarkable difference between the case $K \in (0,1)$ and our situation is that this process is a semimartingale when $2HK = 1$.

1. Introduction

Houdré and Villa in [7] gave the first introduction to the bifractional Brownian motion (bifBm) $B_{H,K} = \left( B_{t}^{H,K}; t \geq 0 \right)$ with parameters $H \in (0,1)$ and $K \in (0,1]$ which is defined as a centered Gaussian process, with covariance function

$$R_{H,K}(t,s) = E \left( B_{t}^{H,K} B_{s}^{H,K} \right) = \frac{1}{2K} \left( (t^{2H} + s^{2H})^{K} - |t-s|^{2HK} \right),$$

for every $s, t \geq 0$.

The case $K = 1$ corresponds to the fractional Brownian motion (fBm) with Hurst parameter $H$. Some properties of the bifractional Brownian motion have been studied by Russo and Tudor in [12]. In fact, in [12] it is shown that the bifractional Brownian motion behaves as a fractional Brownian motion with Hurst parameter $HK$. A stochastic calculus with respect to this process has been recently developed by Kruk, Russo and Tudor [9] and Es-Sebaiy and Tudor [6].

In this paper we prove that, with $H \in (0,1)$ and $HK \in (0,1)$, the process $B_{H,K}$ can be extended for $1 < K < 2$. The case $H = \frac{1}{2}$ and $1 < K < 2$ plays a role to give an extension of sub-fractional Brownian motion (subfBm) (see [4]). The subfBm $(\xi_{h}^{h}, t \geq 0)$ with parameter $0 < h \leq 2$ is a centered Gaussian process with covariance:

$$E \left( \xi_{t}^{h} \xi_{s}^{h} \right) = C_{h} \left( t^{2h} + s^{2h} - \frac{1}{2} ((t+s)2h + |t-s|2h) \right), \quad s, t \geq 0$$

where $C_{h} = 1$ if $0 < h < 1$ and $C_{h} = 2(1-h)$ if $1 < h \leq 2$.

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2. Definition of bifractional Brownian motion with parameter $K \in (1, 2)$

For any $K \in (0, 2)$, let $X^K = (X^K_t, t \geq 0)$ be a Gaussian process defined by

$$X^K_t = \int_0^\infty (1 - e^{-rt})r^{-\frac{1+K}{2}}dW_r, \quad t \geq 0 \quad (2.1)$$

where $(W_t, t \geq 0)$ is a standard Brownian motion.

This process was introduced in [10] for $K \in (0, 1)$ in order to obtain a decomposition of the bifractional Brownian motion with $H \in (0, 1)$ and $K \in (0, 1)$. More precisely, they prove the following result:

**Theorem 2.1** (see [10]). Let $B^{H,K}$ a bifractional Brownian motion with parameters $H \in (0, 1)$ and $K \in (0, 1)$, $B^{HK}$ be a fractional Brownian motion with Hurst parameter $HK \in (0, 1)$ and $W = \{W_t, t \geq 0\}$ a standard Brownian motion. Let $X^K$ be the process given by (2.1). If we suppose that $B^{H,K}$ and $W$ are independents, then the processes $\{Y_t = C_1X^K_{2th} + B^{H,K}_t, t \geq 0\}$ and $\{C_2B^{HK}_t, t \geq 0\}$ have the same distribution, where $C_1 = \sqrt{\frac{2-K}{\Gamma((1-K))}}$ and $C_2 = \frac{2}{2^{1-K}}$.

The process defined in (2.1) has good properties. The following result is proved in [10] for the case $K \in (0, 1)$ and extended to the case $K \in (1, 2)$ in [2] and [11]:

**Proposition 2.2** (see [2],[10] and [11]). The process $X^K = \{X^K_t, t \geq 0\}$ is Gaussian, centered, and its covariance function is:

$$\text{Cov}(X^K_s, X^K_t) = \begin{cases} \Gamma(1-K) \left[t^K + s^K - (t+s)^K\right] & \text{if } K \in (0, 1), \\ \Gamma(2-K) \frac{(t+s)^K - t^K - s^K}{K(K-1)} & \text{if } K \in (1, 2). \end{cases} \quad (2.2)$$

Moreover, $X^K$ has a version with trajectories which are infinitely differentiable on $(0, \infty)$ and absolutely continuous on $[0, \infty)$.

Using the fact that when $K \in (1, 2)$, the covariance function of $X^K$ is given by

$$\text{Cov}(X^K_s, X^K_t) = \frac{\Gamma(2-K)}{K(K-1)} ((t+s)^K - t^K - s^K),$$

and considering also the process

$$X^K_{t,H} = X^K_{2th}; \quad t \geq 0, \quad (2.3)$$

we can prove the following result:

**Theorem 2.3.** Assume $H \in (0, 1)$ and $K \in (1, 2)$ with $HK \in (0, 1)$. Let $B^{HK}$ be a fractional Brownian motion, and $W = \{W_t, t \geq 0\}$ a standard Brownian motion. Let $X^{K,H}$ the process defined in (2.3). If we suppose that $B^{HK}$ and $W$ are independents, then the processes

$$B_{t,H}^K = aB_{t,K}^H + bX_{t,H}^K, \quad (2.4)$$

where $a = \sqrt{2^{-1-K}}$ and $b = \sqrt{\frac{K(K-1)}{2^K(2-K)}}$ is a centered Gaussian process with covariance function

$$E(B_{t,H}^K B_{s,H}^K) = \frac{1}{2^K} \left((2^H + s^H)^K - |t - s|^{2HK}\right); \quad s, \ t \geq 0.$$
Proof. It is obvious that the process defined in (2.4) is a centered Gaussian process. On the other hand, its covariance functions is given by
\[
E \left( B_t^{H,K} B_s^{H,K} \right) = a^2 E \left( B_t^{H,K} B_s^{H,K} \right) + b^2 E \left( X_t^{H,K} X_s^{H,K} \right)
\]
\[
= \frac{1}{2K} \left( (t^{2H} + s^{2H}) - |t-s|^{2H} \right)
\]
\[
+ \frac{1}{2K} \left( \left( (t^{2H} + s^{2H}) - t^{2H} - s^{2H} \right) \right)
\]
\[
= \frac{1}{2K} \left( (t^{2H} + s^{2H}) - |t-s|^{2H} \right),
\]
which completes the proof. □

Thus, the bifractional Brownian motion \( B^{H,K} \) with parameters \( H \in (0,1) \) and \( K \in (1,2) \) such that \( HK \in (0,1) \) is well defined and it has a decomposition as a sum of a fBm \( B^{HK} \) and an absolutely continuous process \( X^{H,K} \).

Remark 2.4. Assume that \( 2HK = 1 \). Russo and Tudor [12] proved that if \( K \) belong to \((0,1)\), the process \( B^{H,K} \) is not a semimartingale. But in the case when \( 1 < K < 2 \), \( B^{H,K} \) is a semimartingale because we have a decomposition of this process as a sum of a Brownian motion \( B^{1/2} \) and a finite variation process \( X^{H,K} \).

The following decomposition is exploited to prove the quasi-helix property (in the sense of J.P. Kahane) of \( B^{H,K} \). This result is satisfied for all \( K \in (0,2) \).

**Proposition 2.5.** Let \( H \in (0,1) \) and \( K \in (0,2) \) such that \( HK \in (0,1) \). Let \( (\xi^{K/2}_t, t \geq 0) \) be a sub-fractional Brownian motion with parameter \( K/2 \in (0,1) \), independent to \( B^{H,K} \) and suppose that \( (B^{K/2}_t, t \geq 0) \) and \( (B^{H,K}_t, t \geq 0) \) are two independent fractional Brownian motions with Hurst parameter \( K/2 \in (0,1) \) and \( HK \in (0,1) \), respectively. We set \( \tilde{\xi}^{K,H}_t = \xi^{K/2}_t \) and \( \tilde{B}^{H,K}_t = B^{K/2}_t, t \geq 0 \). Then, it holds that
\[
B^{H,K} + \sqrt{2^{1-K}} \xi^{K,H}_t \overset{d}{=} \sqrt{2^{1-K}} \left( \tilde{B}^{H,K} + B^{H,K} \right)
\]
where \( \overset{d}{=} \) denotes that both processes have the same distribution.

**Proof.** The result follows easily from the independence and the fact that their corresponding covariance functions satisfy the following equality for all \( s, t \geq 0 \)
\[
R^{H,K}(t,s) = \frac{1}{2K} \left( (t^{2H} + s^{2H}) - |t-s|^{2H} \right)
\]
\[
= 2^{1-K} \left[ -\text{Cov}(\xi^{K,H}_t, \xi^{K,H}_s) + \text{Cov}(\tilde{B}^{H,K}_t, \tilde{B}^{H,K}_s) \right.
\]
\[
+ \text{Cov}(B^{H,K}_t, B^{H,K}_s) \Big] \cdot
\]

**Proposition 2.6.** Let \( H \in (0,1) \) and \( K \in (1,2) \) such that \( HK \in (0,1) \). Then for any \( t, s \geq 0 \), if \( 0 < H \leq 1/2 \)
\[
2^{1-K} |t-s|^{2HK} \leq E \left( B^{H,K}_t - B^{H,K}_s \right)^2 \leq |t-s|^{2HK},
\]
and if $1/2 < H < 1$

$$2^{1-K}|t-s|^{2HK} \leq E\left(B_t^{H,K} - B_s^{H,K}\right)^2 \leq 2^{2-K}|t-s|^{2HK}.$$  

**Proof.** Using the proposition 2.5, we obtain 

$$E\left(B_t^{H,K} - B_s^{H,K}\right)^2 = 2^{1-K}\left(-E\left(\xi_{\epsilon_t}^{\frac{H}{K}} - \xi_{\epsilon_s}^{\frac{H}{K}}\right)^2 + E\left(B_t^{\frac{H}{K}} - B_s^{\frac{H}{K}}\right)^2 + E\left(B_t^{HK} - B_s^{HK}\right)^2\right)$$

$$= 2^{1-K}\left(-E\left(\xi_{\epsilon_t}^{\frac{H}{K}} - \xi_{\epsilon_s}^{\frac{H}{K}}\right)^2 + |t^{2H} - s^{2H}|^K + |t-s|^{2HK}\right).$$

On the other hand, from [3] we have 

$$(2 - 2^{K-1})|t^{2H} - s^{2H}|^K \leq E\left(\xi_{\epsilon_t}^{\frac{H}{K}} - \xi_{\epsilon_s}^{\frac{H}{K}}\right)^2 \leq |t^{2H} - s^{2H}|^K.$$ 

Thus

$$2^{1-K}|t-s|^{2HK} \leq E\left(B_t^{H,K} - B_s^{H,K}\right)^2 \leq 2^{1-K}\left(|t-s|^{2HK} + (2^{K-1} - 1)|t^{2H} - s^{2H}|^K\right).$$

Then we deduce that for every $H \in (0, 1)$, $K \in (1, 2)$ with $HK \in (0, 1)$

$$2^{1-K}|t-s|^{2HK} \leq E\left(B_t^{H,K} - B_s^{H,K}\right)^2$$

and the other hand for every $H \in (0, \frac{1}{2})$, $K \in (1, 2)$ we have

$$E\left(B_t^{H,K} - B_s^{H,K}\right)^2 \leq 2^{1-K}\left(|t-s|^{2HK} + (2^{K-1} - 1)|t^{2H} - s^{2H}|^K\right) \leq |t-s|^{2HK}.$$ 

The last inequality is satisfied from the fact that $|t^{2H} - s^{2H}| \leq |t-s|^{2H}$ for $H \in (0, \frac{1}{2})$.

To complete the proof, it remains to show that for every $H \in (\frac{1}{2}, 1)$, $K \in (1, 2)$ with $HK \in (0, 1)$ (observe that in this situation we have $HK \in (\frac{1}{2}, 1)$)

$$E\left(B_t^{H,K} - B_s^{H,K}\right)^2 \leq 2^{2-K}|t-s|^{2HK}.$$ 

Notice that,

$$E\left(B_t^{H,K} - B_s^{H,K}\right)^2 = \frac{1}{2^K}\left((2t^{2H})^K + (2s^{2H})^K - 2\left((t^{2H} + s^{2H})^K - |t-s|^{2HK}\right)\right)$$

$$= \frac{2}{2^K}|t-s|^{2HK} + \left(t^{2HK} + s^{2HK} - \frac{2}{2^K}(t^{2H} + s^{2H})^K\right).$$

Hence it is enough to prove that

$$t^{2HK} + s^{2HK} - \frac{2}{2^K}(t^{2H} + s^{2H})^K \leq 2^{1-K}|t-s|^{2HK},$$
or equivalently
\[ t^{2HK} + s^{2HK} \leq 2^{1-K} \left( (t^{2H} + s^{2H})^K + |t-s|^{2HK} \right). \]
From now on we will assume, bethought loss of generality, that \( s \leq t \). Dividing by \( t^{2HK} \) we obtain that we have to prove that
\[ 1 + \left( \frac{s}{t} \right)^{2HK} \leq 2^{1-K} \left( 1 + \left( \frac{s}{t} \right)^{2H} \right)^K + \left( 1 - \frac{s}{t} \right)^{2HK}. \]
Equivalently we have to prove that, for any \( u \in (0,1] \) the function
\[ f(u) := 2^{1-K} \left[ (1 + u^{2H})^K + (1-u)^{2HK} \right] - u^{2HK} - 1 \]
is positive.
Observe that \( f(1) = 0 \), so, it is enough to see that the derivative of this function is negative for \( u \in (0,1] \). But,
\[ f'(u) = 2HK2^{1-K}u^{2HK-1} \left[ \left( 1 + \frac{1}{u^{2H}} \right)^{K-1} - \left( \frac{1}{u} - 1 \right)^{2HK-1} - 2^{K-1} \right]. \]
To prove that \( f'(u) \leq 0 \) for \( u \in (0,1] \) it is enough to see that the function
\[ h(u) := \left( 1 + \frac{1}{u^{2H}} \right)^{K-1} - \left( \frac{1}{u} - 1 \right)^{2HK-1} - 2^{K-1}, \]
is negative for \( u \in (0,1] \). But, since \( h(1) = 0 \), it is enough to prove that its derivative \( h'(u) \geq 0 \) for \( u \in (0,1] \). But,
\[ h'(u) = \frac{1}{u^{2HK}} (-2(K-1)H(u^{2H} + 1)^{K-2}u^{2H-1} + (1-u)^{2HK-2}(2HK-1)). \]
Observe that \( u^{2H-1} \leq 1 \) because \( H \in (\frac{1}{2},1) \), \((u^{2H} + 1)^{K-2} \leq 1 \) and \((1-u)^{2HK-2} \geq 1 \).
So,
\[ h'(u) \geq \frac{1}{u^{2HK}} (-2(K-1)H + 2HK - 1) = \frac{1}{u^{2HK}}(2H - 1) \geq 0, \]
because \( H \geq \frac{1}{2} \). The prove is now complete. \( \square \)

**Proposition 2.7.** Suppose that \( H \in (0,1) \), \( K \in (1,2) \) such that \( HK \in (0,1) \). The bifBm \( B^{HK} \) has the following properties

i) \( B^{HK} \) is a self-similar process with index \( HK \), i.e.
\[ (B^{HK}_{a\cdot}, t \geq 0) \overset{d}{=} \left( a^{HK} B^{HK}_t, t \geq 0 \right), \quad \text{for each } a > 0. \]

ii) \( B^{HK} \) has the same long-range property of the fBm \( B^{HK} \), i.e. \( B^{HK} \) has the short-memory for \( HK < \frac{1}{2} \) and it has long-memory for \( HK > \frac{1}{2} \).

iii) \( B^{HK} \) has a \( \frac{1}{HK} \)-variation equals to \( 2\frac{1}{HK} \lambda t \) with \( \lambda = E(|N|^{\frac{1}{HK}}) \) and \( N \) being a standard normal random variable, i.e.
\[ \sum_{j=1}^{n} \left( B^{HK}_{t_j} - B^{HK}_{t_{j-1}} \right)^{\frac{1}{HK}} \overset{n \rightarrow \infty}{\longrightarrow} 2\frac{1}{HK} \lambda t \] \text{ in } L^1(\Omega).

where \( 0 = t_0 < \ldots < t_n = t \) denotes a partition of \([0,t] \).
iv) $B^{H,K}$ is not a semimartingale if $2HK \neq 1$.

The proof of the proposition 2.7 is straightforward from [12] and [6].

3. Space of integrable functions with respect to bifractional Brownian motion

Let us consider $\mathcal{E}$ the set of simple functions on $[0,T]$. Generally, if $U := (U_t, t \in [0,T])$ is a continuous, centered Gaussian process, we denote by $\mathcal{H}_U$ the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = E(U_tU_s).$$

In the case of the standard Brownian motion $W$, the space $\mathcal{H}_W$ is $L^2([0,T])$. On the other hand, for the fractional Brownian motion $B^H$, the space $\mathcal{H}_B$ is the set of restrictions to the space of test functions $D((0,T))$ of the distributions of $W_{1/2}^{-H}, 2((0,T))$ with support contained in $[0,T]$ (see [8]). In the case $H \in (0,1/2)$ all the elements of the domain are functions, and the space $\mathcal{H}_B$ coincides with the fractional Sobolev space $H^{1/2-H}_0(L^2([0,T]))$ (see for instance [5]), but in the case $H \in (1/2, 1)$ this space contains distributions which are not given by any function.

As a direct consequence of Theorem 2.3 we have the following relation between $\mathcal{H}_{B^{H,K}}, \mathcal{H}_{B^{H,K}}$ and $\mathcal{H}_{X^{H,K}}$, where $B^{H,K}$ is the bifractional Brownian motion and $X^{H,K}$ is the process defined in (2.3).

**Proposition 3.1.** Let $H \in (0,1)$ and $K \in (1,2)$ with $HK \in (0,1)$. Then it holds that

$$\mathcal{H}_{X^{H,K}} \cap \mathcal{H}_{B^{H,K}} = \mathcal{H}_{X^{H,K}}.$$

If we consider the processes appearing in Proposition 2.5 we have also the following result:

**Proposition 3.2.** Let $H \in (0,1)$. For every $K \in (0,2)$ with $HK \in (0,1)$ the following equality holds

$$\mathcal{H}_{\xi^{H,K}} \cap \mathcal{H}_{B^{H,K}} = \mathcal{H}_{\xi^{H,K}} \cap \mathcal{H}_{B^{H,K}}.$$ 

**Proof.** Both propositions are a direct consequence of the two decompositions into the sum of two independent processes proved in Theorem 2.3 and Proposition 2.5. \qed

**Remark 3.3.** For the case $K \in (0,1)$ we have the following equality (see [10])

$$\mathcal{H}_{B^{H,K}} = \mathcal{H}_{X^{H,K}} \cap \mathcal{H}_{B^{H,K}}.$$

4. Weak convergence towards the bifractional Brownian motion

Another direct consequence of the decomposition for the bifractional Brownian motion with $H \in (0,1), K \in (1,2)$ and $HK \in (0,1)$ is the following result of convergence in law in the space $C([0,T])$.

Recall that the fractional Brownian motion of Hurst parameter $H \in (0,1)$ admits an integral representation of the form (see for instance [1])

$$B_t^H = \int_0^t K^H(t,s)dW_s,$$
where $W$ is a standard Brownian motion and the kernel $K^H$ is defined on the set \( \{0 < s < t\} \) and given by

\[
K^H(t, s) = d_H(t - s)^{H - \frac{1}{2}} + d_H \int_s^t (u - s)^{H - \frac{1}{2}} \left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2} - H}\right) du,
\]

with \( d_H \) the following normalizing constant

\[
d_H = \left(\frac{2H\Gamma\left(\frac{1}{2} - H\right)}{\Gamma\left(H + \frac{1}{2}\right)\Gamma\left(2 - 2H\right)}\right)^{\frac{1}{2}}.
\]

**Theorem 4.1.** Let \( H \in (0, 1) \) and \( K \in (1, 2) \) with \( HK \in (0, 1) \). Consider \( \theta \in (0, \pi) \cup (\pi, 2\pi) \) such that if \( HK \in (0, \frac{1}{4}] \) then \( \theta \) satisfies that

\[
\cos\left((2i + 1)\theta\right) \neq 1
\]

for all \( i \in \mathbb{N} \) such that \( i \leq \frac{1}{4}\left[H\right] \). Set

\[
a = \sqrt{2^{1-K}} \quad \text{and} \quad b = \sqrt{\frac{K(K-1)}{2^{2-K}}}.\]

Define the processes,

\[
B_{\epsilon}^{HK} = \left\{ \frac{2}{\epsilon} \int_0^T K^{HK}(t, s) \sin\left(\theta N_{\epsilon s}^{\frac{1}{2}}\right) ds, \quad t \in [0, T] \right\},
\]

\[
X_{\epsilon}^{H,K} = \left\{ \frac{2}{\epsilon} \int_0^\infty (1 - e^{-st^{2H}}s^{-\frac{1}{2}+\frac{1}{2H}}) \cos\left(\theta N_{\epsilon s}^{\frac{1}{2}}\right) ds, \quad t \in [0, T] \right\},
\]

where \( K^{HK}(t, s) \) is the kernel defined in (4.1). Then,

\[
\{Y_{\epsilon}^H(t) = aB_{\epsilon}^{HK}(t) + bX_{\epsilon}^{H,K}(t), t \in [0, T]\}
\]

weakly converges in \( C([0, T]) \) to a bifractional Brownian motion.

**Proof.** Applying Theorems 3.2 and 3.5 of [2] we know that, respectively, the processes \( B_{\epsilon}^{HK} \) and \( X_{\epsilon}^{H,K} \) converge in law in \( C([0, T]) \) towards a fBm \( B^{HK} \) and to the process \( X^{H,K} \). Moreover, applying Theorem 2.1 of [2], we know that the limit laws are independent. Hence, we are under the hypothesis of the decomposition obtained in Theorem 2.3, which proves the stated result. \( \square \)

**Remark 4.2.** Obviously we can also obtain the same result interchanging the roles of the sinus and the cosine functions in the definition of the approximating processes.

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