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Estimation of the drift of fractional Brownian motion

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Abstract

We consider the problem of efficient estimation for the drift of fractional Brownian motion $B^H := (B_t^H)_{t \in [0,T]}$ with hurst parameter H less than $\frac{1}{2}$. We also construct superefficient James-Stein type estimators which dominate, under the usual quadratic risk, the natural maximum likelihood estimator.

Key words : Fractional Brownian Motion, Stein estimate, MLE

2000 Mathematics Subject Classification: 60G15, 62G05, 62B05, 62M09.

1 Introduction

Fix $H \in (0, 1)$ and T > 0. Let $B^H = \left\{ (B_t^{H,1}, ..., B_t^{H,d}); t \in [0, T] \right\}$ be a *d*-dimensional fractional Brownian motion (fBm) defined on the probability space (Ω, \mathcal{F}, P) . That is, B^H is a zero mean Gaussian vector whose components are independent one-dimensional fractional Brownian motions with Hurst parameter $H \in (0, 1)$, i.e., for every $i = 1, ..., d B^{H,i}$ is a Gaussian process and covariance function given by

$$E(B_s^{H,i}B_t^{H,i}) = \frac{1}{2} \left(s^{2H} + t^{2H} - |t-s|^{2H} \right), \quad s,t \in [0,T].$$

For each i = 1, ..., d, $(\mathcal{F}_t^i)_{t \in [0,T]}$ denotes the filtration generated by $(B_t^{H,i})_{t \in [0,T]}$. The fBm was first introduced by [5] and studied by [6]. Notice that if $H = \frac{1}{2}$, the

1

process $B^{\frac{1}{2}}$ is a standard Brownian motion. However, for $H \neq \frac{1}{2}$, the fBm is neither a Markov process, nor a semi-martingale.

Let M be a subspace of the Cameron-Martin space defined by

$$\begin{split} M &= \left\{ \varphi: [0,T] \to \mathbb{R}^d; \varphi_t^i = \int_0^t \dot{\varphi}_s^i ds \text{ with } \dot{\varphi}^i \in L^2([0,T]) \\ \text{and } \varphi^i \in I_{0^+}^{H+\frac{1}{2}} \left(L^2([0,T]) \right), i = 1, ..., d \right\}. \end{split}$$

Let $\theta = \{(\theta_t^1, \ldots, \theta_t^d); t \in [0, T]\}$ be a function belonging to M. Then, Applying Girsanov theorem (see Theorem 2 in [9]), there exist a probability measure absolutely continuous with respect to P under which the process \tilde{B}^H defined by

$$\widetilde{B}_t^H = B_t^H - \theta_t, \qquad t \in [0, T]$$
(1.1)

is a fBm with Hurst parameter H and mean zero. In this case, we say that, under the probability P_{θ} , the process B^{H} is a fBm with drift θ .

We consider in this paper the problem of estimating the drift θ of B^H under the probability P_{θ} , with hurst parameter H < 1/2. We wish to estimate θ under the usual quadratic risk, that is defined for any estimator δ of θ by

$$\mathcal{R}(\theta, \delta) = E_{\theta} \left[\int_{0}^{T} ||\delta_{t} - \theta_{t}||^{2} dt \right]$$

where E_{θ} is the expectation with respect to a probability P_{θ} .

Let $X = (X^1, \ldots, X^d)$ be a normal vector with mean $\theta = (\theta^1, \ldots, \theta^d) \in \mathbb{R}^d$ and identity covariance matrix $\sigma^2 I_d$. The usual maximum likelihood estimator of θ is X itself. Moreover, it is efficient in the sense that the Cramer-Rao bound over all unbiased estimators is attained by X. That is

$$\sigma^2 d = E\left[\|X - \theta\|_d^2 \right] = \inf_{\xi \in \mathcal{S}} E\left[\|\xi - \theta\|_d^2 \right]$$

where S is the class of unbiased estimators of θ and $\|.\|_d$ denotes the Euclidean norm on \mathbb{R}^d .

[12] constructed biased superefficient estimators of θ of the form

$$\delta_{a,b}(X) = \left(1 - \frac{b}{a + ||X||^2}\right) X$$

for a sufficiently small and b sufficiently large when $d \ge 3$. [4] sharpened later this result and presented an explicit class of biased superefficient estimators of the form

$$\left(1 - \frac{a}{||X||_d^2}\right) X$$
, for $0 < a < 2(d-2)$.

Recently, an infinite-dimensional extension of this result has been given by [10]. The authors constructed unbiased estimators of the drift $(\theta_t)_{t\in[0,T]}$ of a continuous Gaussian martingale $(X_t)_{t\in[0,T]}$ with quadratic variation $\sigma_t^2 dt$, where $\sigma \in L^2([0,T], dt)$ is an a.e. non-vanishing function. More precisely, they proved that $\hat{\theta} = (X_t)_{t\in[0,T]}$ is an efficient estimator of $(\theta_t)_{t\in[0,T]}$. On the other hand, using Malliavin calculus, they constructed superefficient estimators for the drift of a Gaussian processe of the form:

$$X_t := \int_0^t K(t,s) dW_s, \qquad t \in [0,T],$$

where $(W_t)_{t \in [0,T]}$ is a standard Brownian motion and K(.,.) is a deterministic kernel. These estimators are biased and of the form $X_t + D_t \log F$, where F is a positive superharmonic random variable and D is the Malliavin derivative.

In Section 3, we prove, using technic based on the fractional calculus and Girsanov theorem, that $\hat{\theta} = B^H$ is an efficient estimator of θ under the probability P_{θ} with risk

$$\mathcal{R}(\theta, B^H) = E_{\theta} \left[\int_0^T \|B_t^H - \theta_t\|^2 dt \right] = \frac{T^{2H+1}}{2H+1} dt$$

Moreover, we will establish that $\hat{\theta} = B^H$ is a maximum likelihood estimator of θ .

In Section 4, we construct a class of biased estimators of James-Stein type of the form

$$\delta(B^H)_t = \left(1 - at^{2H} \left(\frac{r(\|B_t^H\|^2)}{\|B_t^H\|^2}\right)\right) B_t^H, \qquad t \in [0, T].$$

We give sufficient conditions on the function r and on the constant a in order that $\delta(B^H)$ dominates B^H under the usual quadratic risk i.e.

$$\mathcal{R}\left(\theta, \delta\left(B^{H}\right)\right) < \mathcal{R}\left(\theta, B^{H}\right) \qquad \text{for all } \theta \in M.$$
(1.2)

2 Preliminaries

This section contains the elements from fractional calculus that we will need in the paper.

The fractional Brownian motion B^H has the following stochastic integral representation (see for instance, [1], [8])

$$B_t^{H,i} = \int_0^t K_H(t,s) dW_s^i, \quad i = 1, ..., d; \quad t \in [0,T]$$
(2.3)

where $W = (W^1, ..., W^d)$ denotes the d-dimensional Brownian motion and the kernel $K_H(t, s)$ is equal to

$$c_{H}(t-s)^{H-\frac{1}{2}} + c_{H}(\frac{1}{2}-H) \int_{s}^{t} (u-s)^{H-\frac{3}{2}} \left(1-(\frac{s}{u})^{\frac{1}{2}-H}\right) du \quad \text{if } H \le \frac{1}{2}$$
$$c_{H}(H-\frac{1}{2}) \int_{s}^{t} (u-s)^{H-\frac{3}{2}} \left(\frac{s}{u}\right)^{H-\frac{1}{2}} du \quad \text{if } H > \frac{1}{2},$$

if s < t and $K_H(t, s) = 0$ if $s \ge t$. Here c_H is the normalizing constant

$$c_H = \sqrt{\frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2})\Gamma(2 - 2H)}}$$

where Γ is the Euler function.

We recall some basic definitions and results on classical fractional calculus which we will need. General information about fractional calculus can be found in [11]. The left fractional Riemann-Liouville integral of $f \in L^1((a, b))$ of order $\alpha > 0$ on (a, b) is given at almost all $x \in (a, b)$ by

$$I_{a^+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy.$$

If $f \in I^{\alpha}_{a^+}(L^p(a, b))$ with $0 < \alpha < 1$ and p > 1 then the left-sided Riemann-Liouville derivative of f of order α defined by

$$D_{a^+}^{\alpha}f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy\right)$$

for almost all $x \in (a, b)$. For $H \in (0, 1)$, the integral transform

$$(K_H f)(t) = \int_0^t K_H(t,s)f(s)ds$$

is a isomorphism from $L^2([0,1])$ onto $I_{0^+}^{H+\frac{1}{2}}(L^2([0,1]))$ and its inverse operator K_H^{-1} is given by

$$K_{H}^{-1}f = t^{H-\frac{1}{2}}D_{0^{+}}^{H-\frac{1}{2}}t^{\frac{1}{2}-H}f' \quad \text{for } H > 1/2,$$
(2.4)

$$K_{H}^{-1}f = t^{\frac{1}{2}-H}D_{0^{+}}^{\frac{1}{2}-H}t^{H-\frac{1}{2}}D_{0^{+}}^{2H}f \quad \text{for } H < 1/2.$$
(2.5)

Moreover, for $H < \frac{1}{2}$, if f is an absolutely continuous function then $K_H^{-1}f$ can be represented of the form (see [9])

$$K_{H}^{-1}f = t^{H-\frac{1}{2}}I_{0^{+}}^{\frac{1}{2}-H}t^{\frac{1}{2}-H}f'.$$
(2.6)

3 The maximum likelihood estimator and Cramer-Rao type bound

We consider a function $\theta = (\theta^1, \dots, \theta^d)$ belonging to M. An estimator $\xi = (\xi^1, \dots, \xi^d)$ of $\theta = (\theta^1, \dots, \theta^d)$ is called unbiased if, for every $t \in [0, T]$

$$E_{\theta}(\xi_t^i) = \theta_t^i, \qquad i = 1, \dots, d$$

and it is called adapted if, for each i = 1, ..., d, ξ^i is adapted to $(\mathcal{F}_t^i)_{t \in [0,T]}$. Since for any i = 1, ..., d, the function θ^i is deterministic and

$$\int_0^T (K_H^{-1}(\theta^i)(s))^2 ds < \infty,$$

then Girsanov theorem yields that there exists a probability measure P_{θ} absolutely continuous with respect to P under which the process $\widetilde{B}^{H} := \left(\widetilde{B}_{t}^{H}; t \in [0, T]\right)$ defined by

$$\widetilde{B}_t^H = B_t^H - \theta_t, \qquad t \in [0, T]$$
(3.7)

is a d-dimensional fBm with Hurst parameter H and mean zero. Moreover the Girsanov density of P_{θ} with respect to P is given by:

$$\frac{dP_{\theta}}{dP} = \exp\left[\sum_{i=1}^{d} \left(\int_{0}^{T} K_{H}^{-1}(\theta^{i})(s) dW_{s}^{i} - \frac{1}{2} \int_{0}^{T} (K_{H}^{-1}(\theta^{i})(s))^{2} ds\right)\right]$$

and

$$\widetilde{B}_t^H = \int_0^t K_H(t,s) d\widetilde{W}_s$$

where \widetilde{W} is a d-dimensional Brownian motion under the probability P_{θ} and

$$\widetilde{W}_{t}^{i} = W_{t}^{i} - \int_{0}^{t} K_{H}^{-1}(\theta^{i})(s) ds, \quad i = 1, ..., d; \quad t \in [0, T].$$

The equation (3.7) implies that B^H is an unbiased and adapted estimator of θ under probability P_{θ} . In addition, we obtain the Cramer-Rao type bound:

$$R(H,\hat{\theta}) := \mathcal{R}(\theta, B^H) = \int_0^T E_{\theta} \|\widetilde{B}_t^H\|^2 dt = d \int_0^T t^{2H} dt = \frac{T^{2H+1}}{2H+1} d.$$

The first main result of this section is given by the following proposition which asserts that $\hat{\theta} = B^H$ is an efficient estimator of θ .

Theorem 1 Assume that $H < \frac{1}{2}$. If ξ is an unbiased and adapted estimator of θ , then

$$E_{\theta} \int_0^T \|\xi_t - \theta_t\|^2 dt \ge R(H, \hat{\theta}).$$
(3.8)

Proof: Since ξ is unbiased, then for every $\varphi \in M$ we have

$$E_{\varphi}(\xi_t^j) = E_{\varphi}(\varphi_t^j), \qquad j = 1, \dots, d.$$

Let $\varphi = \theta + \varepsilon \psi$ with $\psi \in M$ and $\varepsilon \in \mathbb{R}$. Then for every $t \in [0, T]$ and $j \in \{1, \dots, d\}$, we have

$$E_{\theta+\varepsilon\psi}(\xi_t^j) = E_{\theta+\varepsilon\psi}(\theta_t^j + \varepsilon\psi_t^j)$$
$$= E_{\theta+\varepsilon\psi}(\theta_t^j) + \varepsilon\psi_t^j.$$

 $\mathbf{6}$

This implies that for every $j = 1, \ldots, d$

Applying Cauchy-Schwarz inequality in $L^2(\Omega, dP_{\theta})$, we obtain that for every $t \in [0, T]$

$$\begin{aligned} \|\psi_t\|^2 &= \sum_{j=1}^d (\psi_t^j)^2 &\leq \sum_{j=1}^d E_\theta \left((\xi_t^j - \theta_t^j)^2 \right) E_\theta \left(\left[\int_0^t K_H^{-1}(\psi^j)(s) d\widetilde{W}_s^j \right]^2 \right) \\ &= \sum_{j=1}^d E_\theta \left[\left((\xi_t^j - \theta_t^j)^2 \right) \int_0^t (K_H^{-1}(\psi^j)(s))^2 ds \right]. \end{aligned}$$

We take for each j = 1, ..., d, $\psi_t^j = t^{2H}$. Since $t \longrightarrow t^{2H}$ is absolutely continuous function, then by (2.6), a simple calculation shows that

$$\begin{split} K_{H}^{-1}(t^{2H}) &= 2Ht^{H-\frac{1}{2}}I_{0^{+}}^{\frac{1}{2}-H}t^{H-\frac{1}{2}} \\ &= \frac{2H\beta(\frac{1}{2}-H,H+\frac{1}{2})}{\Gamma(\frac{1}{2}-H)}t^{H-1/2} \\ &= 2H(\Gamma(\frac{1}{2}+H))t^{H-1/2}. \end{split}$$

It is known that

$$0 \le \Gamma(z) \le 1 \quad \text{for every } z \in [1, 2]. \tag{3.9}$$

Combining the facts that $z\Gamma(z) = \Gamma(z+1), z > 0, 2H \le (H+\frac{1}{2})^2$ and (3.9), we obtain

$$dt^{2H} = \|\psi_t\|^2 \leq (\Gamma(\frac{3}{2} + H))^2 E_\theta \left(\|\xi_t - \theta_t\|^2\right) \leq E_\theta \left(\|\xi_t - \theta_t\|^2\right).$$

Hence, by an integration with respect to dt, we get

$$R(H,\hat{\theta}) = \frac{T^{2H+1}}{2H+1} \le E_{\theta} \int_{0}^{T} \|\xi_{t} - \theta_{t}\|^{2} dt.$$

Therefore (3.8) is satisfied.

Corollary 1 The process $\hat{\theta} = B^H$ is a maximum likelihood estimator of θ .

Proof: We have for every $\psi \in M$

$$\frac{d}{d\varepsilon}_{\varepsilon=0} \exp\left[\sum_{i=1}^{d} \int_{0}^{t} K_{H}^{-1}(\hat{\theta}^{i} + \varepsilon\psi^{i})(s) dW_{s}^{i} - \frac{1}{2} \int_{0}^{t} (K_{H}^{-1}(\hat{\theta}^{i} + \varepsilon\psi^{i})(s))^{2}) ds\right] = 0.$$

Hence

$$\sum_{i=1}^{d} \left(\int_{0}^{t} K_{H}^{-1}(\psi^{i})(s) dW_{s}^{i} - \int_{0}^{t} K_{H}^{-1}(\psi^{i})(s) K_{H}^{-1}(\hat{\theta}^{i})(s) ds \right) = 0.$$

Which implies that for every i = 1, ..., d

$$E\left(\int_0^t K_H^{-1}(\psi^i)(s)dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s)K_H^{-1}(\hat{\theta}^i)(s)ds\right)^2 = 0.$$

Then, for each i = 1, ..., d

$$W_t^i = \int_0^t K_H^{-1}(\hat{\theta}^i)(s) ds, \qquad t \in [0, T].$$

Therefore by (2.3), we obtain that $B^H = \hat{\theta}$.

4 Superefficient James-Stein type estimators

The aim of this section is to construct superefficient estimators of θ which dominate, under the usual quadratic risk, the natural MLE estimator B^{H} . The class of estimators considered here are of the form

$$\delta(B^{H})_{t} = B_{t}^{H} + g(B_{t}^{H}, t), \qquad t \in [0, T]$$
(4.10)

where $g: \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^d$ is a function. The problem turns to find sufficient conditions on g such that $\mathcal{R}\left(\theta, \delta(B^H)\right) < \infty$ and the risk difference is negative, i.e.

$$\Delta \mathcal{R}(\theta) = \mathcal{R}\left(\theta, \delta(B^H)\right) - \mathcal{R}\left(\theta, B^H\right) < 0.$$

In the sequel we assume that the function g satisfies the following assumption:

$$(A) \begin{cases} E_{\theta} \left[\int_{0}^{T} ||g(B_{t}^{H}, t)||_{d}^{2} dt \right] < \infty, \\ \text{the partial derivatives } \partial_{i}g^{i} := \frac{\partial g^{i}}{\partial x^{i}}, \ i = 1, \dots, n \text{ of } g \text{ exist.} \end{cases}$$

Then $\mathcal{R}\left(\theta, \delta(B^H)\right) < \infty$. Moreover

$$\begin{aligned} \Delta \mathcal{R}(\theta) &= E_{\theta} \left[\int_{0}^{T} ||B_{t}^{H} + g(B_{t}^{H}, t) - \theta_{t}||_{d}^{2} - ||B_{t}^{H} - \theta_{t}||_{d}^{2} dt \right] \\ &= E_{\theta} \left[\int_{0}^{T} ||g(B_{t}^{H}, t)||_{d}^{2} + 2\sum_{i=1}^{d} \left(g^{i}(B_{t}^{H}, t)(B_{t}^{H,i} - \theta_{t}^{i}) \right) dt \right]. \end{aligned}$$

In addition,

$$E_{\theta} \int_{0}^{T} \sum_{i=1}^{d} \left(g^{i}(B_{t}^{H}, t)(B_{t}^{H,i} - \theta_{t}^{i}) \right) dt$$

$$= \sum_{i=1}^{d} \int_{0}^{T} (2\pi t^{2H})^{-\frac{d}{2}} \left(\int_{\mathbb{R}^{d}} g^{i}(x^{1}, \dots, x^{d}, t)(x^{i} - \theta_{t}^{i}) \times e^{-\frac{\sum_{j=1}^{d} (x^{j} - \theta_{t}^{j})^{2}}{2t^{2H}}} dx^{1} \dots dx^{d} \right) dt$$

$$= \sum_{i=1}^{d} \int_{0}^{T} (2\pi t^{2H})^{-\frac{d}{2}} \left(\int_{\mathbb{R}^{d}} t^{2H} \partial_{i} g^{i}(x^{1}, \dots, x^{d}, t) \times e^{-\frac{\sum_{j=1}^{d} (x^{j} - \theta_{t}^{j})^{2}}{2t^{2H}}} dx^{1} \dots dx^{d} \right) dt$$

$$= \sum_{i=1}^{d} \int_{0}^{T} \left(t^{2H} E_{\theta} \partial_{i} g^{i}(B_{t}^{H}, t) \right) dt = E_{\theta} \left[\sum_{i=1}^{d} \int_{0}^{T} t^{2H} \partial_{i} g^{i}(B_{t}^{H}, t) dt \right].$$

Consequently, the risk difference equals

$$\Delta \mathcal{R}(\theta) = E_{\theta} \left[\int_{0}^{T} \left(||g(B_{t}^{H}, t)||^{2} + 2t^{2H} \sum_{i=1}^{d} \partial_{i}g^{i}(B_{t}^{H}, t) \right) dt \right].$$
(4.11)

We can now state the following theorem.

Theorem 2 Let $g : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^d$ be a function satisfying (A). A sufficient conditions for the estimator $(B_t^H + g(B_t^H, t))_{t \in [0,T]}$ to dominate B^H under the usual quadratic risk is

$$E_{\theta} \left[\int_{0}^{T} \left(||g(B_{t}^{H}, t)||^{2} + 2t^{2H} \sum_{i=1}^{d} \partial_{i}g^{i}(B_{t}^{H}, t) \right) dt \right] < 0$$

As an application, take g of the form

$$g(x,t) = at^{2H} \frac{r(\|x\|^2)}{\|x\|^2} x,$$
(4.12)

where a is a constant strictly positive and $r : \mathbb{R}^+ \to \mathbb{R}^+$ is bounded derivable function. The next lemma give a sufficient condition for g in (4.12) to satisfies the assumption (A).

Lemma 1 If $d \ge 3$ and $H < \frac{1}{2}$ then

$$E\left[\int_0^T \frac{1}{\|B_t^H\|^2} dt\right] < \infty.$$

$$(4.13)$$

Proof: Firstly the integral given by (4.13) is well defined, because

$$(dt \times P)\left((t, w); B_t^H(w) = 0\right) = 0$$

where $(dt \times P)$ is the product measure.

Using the change of variable and $d \ge 3$ we see that

$$E\int_0^T \frac{1}{\|B_t^H\|^2} dt = \int_0^T \frac{dt}{t^{2H}} \int_{\mathbb{R}^d} \frac{e^{-\frac{\|y\|^2}{2}}}{\sqrt{2\pi} \|y\|^2} dy \le C\int_0^T \frac{1}{t^{2H}} dt$$

where C is a constant depending only on d. Furthermore, since $H < \frac{1}{2}$ then (4.13) holds.

Theorem 3 Assume that $d \ge 3$. If the function r, the constant a and the parameter H satisfy:

- *i*) $0 \le r(.) \le 1$
- ii) $r(\cdot)$ is differentiable and increasing

iii) $0 < a \le 2(d-2)$ and H < 1/2,

then the estimator

$$\delta(B^H) = B_t^H - at^{2H} \frac{r\left(\|B_t^H\|^2\right)}{\|B_t^H\|^2} B_t^H, \qquad t \in [0, T].$$

dominates B^H .

Proof: It suffices to prove that $\Delta \mathcal{R}(\theta) < 0$. From (4.11) and the hypothesis *i*) and *ii*) we can write

Combining this fact with the assumption *iii*) yields that the risk difference is negative. Which proves the desired result.

For r = 1, we obtain a James-Stein type estimator:

Corollary 2 Let $d \ge 3$, $0 < H < \frac{1}{2}$ and $0 < a \le 2(d-2)$. Then the estimator

$$\left(1 - \frac{at^{2H}}{\|B_t^H\|^2}\right) B_t^H, \qquad t \in [0, T]$$

dominates B^H .

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11

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12