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# Estimation of the drift of fractional Brownian motion

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## Abstract

We consider the problem of efficient estimation for the drift of fractional Brownian motion  $B^H := (B_t^H)_{t \in [0, T]}$  with Hurst parameter  $H$  less than  $\frac{1}{2}$ . We also construct superefficient James-Stein type estimators which dominate, under the usual quadratic risk, the natural maximum likelihood estimator.

**Key words :** Fractional Brownian Motion, Stein estimate, MLE

**2000 Mathematics Subject Classification:** 60G15, 62G05, 62B05, 62M09.

## 1 Introduction

Fix  $H \in (0, 1)$  and  $T > 0$ . Let  $B^H = \{(B_t^{H,1}, \dots, B_t^{H,d}); t \in [0, T]\}$  be a  $d$ -dimensional fractional Brownian motion (fBm) defined on the probability space  $(\Omega, \mathcal{F}, P)$ . That is,  $B^H$  is a zero mean Gaussian vector whose components are independent one-dimensional fractional Brownian motions with Hurst parameter  $H \in (0, 1)$ , i.e., for every  $i = 1, \dots, d$   $B^{H,i}$  is a Gaussian process and covariance function given by

$$E(B_s^{H,i} B_t^{H,i}) = \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H}), \quad s, t \in [0, T].$$

For each  $i = 1, \dots, d$ ,  $(\mathcal{F}_t^i)_{t \in [0, T]}$  denotes the filtration generated by  $(B_t^{H,i})_{t \in [0, T]}$ .

The fBm was first introduced by [5] and studied by [6]. Notice that if  $H = \frac{1}{2}$ , the

process  $B^{\frac{1}{2}}$  is a standard Brownian motion. However, for  $H \neq \frac{1}{2}$ , the fBm is neither a Markov process, nor a semi-martingale.

Let  $M$  be a subspace of the Cameron-Martin space defined by

$$M = \left\{ \varphi : [0, T] \rightarrow \mathbb{R}^d; \varphi_t^i = \int_0^t \dot{\varphi}_s^i ds \text{ with } \dot{\varphi}^i \in L^2([0, T]) \right. \\ \left. \text{and } \varphi^i \in I_{0+}^{H+\frac{1}{2}}(L^2([0, T])), i = 1, \dots, d \right\}.$$

Let  $\theta = \{(\theta_t^1, \dots, \theta_t^d); t \in [0, T]\}$  be a function belonging to  $M$ . Then, Applying Girsanov theorem (see Theorem 2 in [9]), there exist a probability measure absolutely continuous with respect to  $P$  under which the process  $\tilde{B}^H$  defined by

$$\tilde{B}_t^H = B_t^H - \theta_t, \quad t \in [0, T] \quad (1.1)$$

is a fBm with Hurst parameter  $H$  and mean zero. In this case, we say that, under the probability  $P_\theta$ , the process  $B^H$  is a fBm with drift  $\theta$ .

We consider in this paper the problem of estimating the drift  $\theta$  of  $B^H$  under the probability  $P_\theta$ , with hurst parameter  $H < 1/2$ . We wish to estimate  $\theta$  under the usual quadratic risk, that is defined for any estimator  $\delta$  of  $\theta$  by

$$\mathcal{R}(\theta, \delta) = E_\theta \left[ \int_0^T \|\delta_t - \theta_t\|^2 dt \right]$$

where  $E_\theta$  is the expectation with respect to a probability  $P_\theta$ .

Let  $X = (X^1, \dots, X^d)$  be a normal vector with mean  $\theta = (\theta^1, \dots, \theta^d) \in \mathbb{R}^d$  and identity covariance matrix  $\sigma^2 I_d$ . The usual maximum likelihood estimator of  $\theta$  is  $X$  itself. Moreover, it is efficient in the sense that the Cramer-Rao bound over all unbiased estimators is attained by  $X$ . That is

$$\sigma^2 d = E [\|X - \theta\|_d^2] = \inf_{\xi \in \mathcal{S}} E [\|\xi - \theta\|_d^2],$$

where  $\mathcal{S}$  is the class of unbiased estimators of  $\theta$  and  $\|\cdot\|_d$  denotes the Euclidean norm on  $\mathbb{R}^d$ .

[12] constructed biased superefficient estimators of  $\theta$  of the form

$$\delta_{a,b}(X) = \left( 1 - \frac{b}{a + \|X\|^2} \right) X$$

for  $a$  sufficiently small and  $b$  sufficiently large when  $d \geq 3$ . [4] sharpened later this result and presented an explicit class of biased superefficient estimators of the form

$$\left(1 - \frac{a}{\|X\|_d^2}\right) X, \text{ for } 0 < a < 2(d-2).$$

Recently, an infinite-dimensional extension of this result has been given by [10]. The authors constructed unbiased estimators of the drift  $(\theta_t)_{t \in [0, T]}$  of a continuous Gaussian martingale  $(X_t)_{t \in [0, T]}$  with quadratic variation  $\sigma_t^2 dt$ , where  $\sigma \in L^2([0, T], dt)$  is an a.e. non-vanishing function. More precisely, they proved that  $\hat{\theta} = (X_t)_{t \in [0, T]}$  is an efficient estimator of  $(\theta_t)_{t \in [0, T]}$ . On the other hand, using Malliavin calculus, they constructed superefficient estimators for the drift of a Gaussian process of the form:

$$X_t := \int_0^t K(t, s) dW_s, \quad t \in [0, T],$$

where  $(W_t)_{t \in [0, T]}$  is a standard Brownian motion and  $K(\cdot, \cdot)$  is a deterministic kernel. These estimators are biased and of the form  $X_t + D_t \log F$ , where  $F$  is a positive superharmonic random variable and  $D$  is the Malliavin derivative.

In Section 3, we prove, using technic based on the fractional calculus and Girsanov theorem, that  $\hat{\theta} = B^H$  is an efficient estimator of  $\theta$  under the probability  $P_\theta$  with risk

$$\mathcal{R}(\theta, B^H) = E_\theta \left[ \int_0^T \|B_t^H - \theta_t\|^2 dt \right] = \frac{T^{2H+1}}{2H+1} d.$$

Moreover, we will establish that  $\hat{\theta} = B^H$  is a maximum likelihood estimator of  $\theta$ .

In Section 4, we construct a class of biased estimators of James-Stein type of the form

$$\delta(B^H)_t = \left(1 - at^{2H} \left(\frac{r(\|B_t^H\|^2)}{\|B_t^H\|^2}\right)\right) B_t^H, \quad t \in [0, T].$$

We give sufficient conditions on the function  $r$  and on the constant  $a$  in order that  $\delta(B^H)$  dominates  $B^H$  under the usual quadratic risk i.e.

$$\mathcal{R}(\theta, \delta(B^H)) < \mathcal{R}(\theta, B^H) \quad \text{for all } \theta \in M. \quad (1.2)$$

## 2 Preliminaries

This section contains the elements from fractional calculus that we will need in the paper.

The fractional Brownian motion  $B^H$  has the following stochastic integral representation (see for instance, [1], [8])

$$B_t^{H,i} = \int_0^t K_H(t,s) dW_s^i, \quad i = 1, \dots, d; \quad t \in [0, T] \quad (2.3)$$

where  $W = (W^1, \dots, W^d)$  denotes the  $d$ -dimensional Brownian motion and the kernel  $K_H(t, s)$  is equal to

$$\begin{aligned} c_H(t-s)^{H-\frac{1}{2}} + c_H\left(\frac{1}{2}-H\right) \int_s^t (u-s)^{H-\frac{3}{2}} \left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right) du & \quad \text{if } H \leq \frac{1}{2} \\ c_H\left(H - \frac{1}{2}\right) \int_s^t (u-s)^{H-\frac{3}{2}} \left(\frac{s}{u}\right)^{H-\frac{1}{2}} du & \quad \text{if } H > \frac{1}{2}, \end{aligned}$$

if  $s < t$  and  $K_H(t, s) = 0$  if  $s \geq t$ . Here  $c_H$  is the normalizing constant

$$c_H = \sqrt{\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}}$$

where  $\Gamma$  is the Euler function.

We recall some basic definitions and results on classical fractional calculus which we will need. General information about fractional calculus can be found in [11].

The left fractional Riemann-Liouville integral of  $f \in L^1((a, b))$  of order  $\alpha > 0$  on  $(a, b)$  is given at almost all  $x \in (a, b)$  by

$$I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy.$$

If  $f \in I_{a+}^\alpha(L^p(a, b))$  with  $0 < \alpha < 1$  and  $p > 1$  then the left-sided Riemann-Liouville derivative of  $f$  of order  $\alpha$  defined by

$$D_{a+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} dy \right)$$

for almost all  $x \in (a, b)$ .

For  $H \in (0, 1)$ , the integral transform

$$(K_H f)(t) = \int_0^t K_H(t,s) f(s) ds$$

is a isomorphism from  $L^2([0, 1])$  onto  $I_{0+}^{H+\frac{1}{2}}(L^2([0, 1]))$  and its inverse operator  $K_H^{-1}$  is given by

$$K_H^{-1} f = t^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} t^{\frac{1}{2}-H} f' \quad \text{for } H > 1/2, \quad (2.4)$$

$$K_H^{-1} f = t^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H} t^{H-\frac{1}{2}} D_{0+}^{2H} f \quad \text{for } H < 1/2. \quad (2.5)$$

Moreover, for  $H < \frac{1}{2}$ , if  $f$  is an absolutely continuous function then  $K_H^{-1} f$  can be represented of the form ( see [9] )

$$K_H^{-1} f = t^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} t^{\frac{1}{2}-H} f'. \quad (2.6)$$

### 3 The maximum likelihood estimator and Cramer-Rao type bound

We consider a function  $\theta = (\theta^1, \dots, \theta^d)$  belonging to  $M$ . An estimator  $\xi = (\xi^1, \dots, \xi^d)$  of  $\theta = (\theta^1, \dots, \theta^d)$  is called unbiased if, for every  $t \in [0, T]$

$$E_\theta(\xi_t^i) = \theta_t^i, \quad i = 1, \dots, d$$

and it is called adapted if, for each  $i = 1, \dots, d$ ,  $\xi^i$  is adapted to  $(\mathcal{F}_t^i)_{t \in [0, T]}$ .

Since for any  $i = 1, \dots, d$ , the function  $\theta^i$  is deterministic and

$$\int_0^T (K_H^{-1}(\theta^i)(s))^2 ds < \infty,$$

then Girsanov theorem yields that there exists a probability measure  $P_\theta$  absolutely continuous with respect to  $P$  under which the process  $\tilde{B}^H := (\tilde{B}_t^H; t \in [0, T])$  defined by

$$\tilde{B}_t^H = B_t^H - \theta_t, \quad t \in [0, T] \quad (3.7)$$

is a d-dimensional fBm with Hurst parameter  $H$  and mean zero. Moreover the Girsanov density of  $P_\theta$  with respect to  $P$  is given by:

$$\frac{dP_\theta}{dP} = \exp \left[ \sum_{i=1}^d \left( \int_0^T K_H^{-1}(\theta^i)(s) dW_s^i - \frac{1}{2} \int_0^T (K_H^{-1}(\theta^i)(s))^2 ds \right) \right]$$

and

$$\widetilde{B}_t^H = \int_0^t K_H(t, s) d\widetilde{W}_s$$

where  $\widetilde{W}$  is a  $d$ -dimensional Brownian motion under the probability  $P_\theta$  and

$$\widetilde{W}_t^i = W_t^i - \int_0^t K_H^{-1}(\theta^i)(s) ds, \quad i = 1, \dots, d; \quad t \in [0, T].$$

The equation (3.7) implies that  $B^H$  is an unbiased and adapted estimator of  $\theta$  under probability  $P_\theta$ . In addition, we obtain the Cramer-Rao type bound:

$$R(H, \hat{\theta}) := \mathcal{R}(\theta, B^H) = \int_0^T E_\theta \|\widetilde{B}_t^H\|^2 dt = d \int_0^T t^{2H} dt = \frac{T^{2H+1}}{2H+1} d.$$

The first main result of this section is given by the following proposition which asserts that  $\hat{\theta} = B^H$  is an efficient estimator of  $\theta$ .

**Theorem 1** *Assume that  $H < \frac{1}{2}$ . If  $\xi$  is an unbiased and adapted estimator of  $\theta$ , then*

$$E_\theta \int_0^T \|\xi_t - \theta_t\|^2 dt \geq R(H, \hat{\theta}). \quad (3.8)$$

*Proof:* Since  $\xi$  is unbiased, then for every  $\varphi \in M$  we have

$$E_\varphi(\xi_t^j) = E_\varphi(\varphi_t^j), \quad j = 1, \dots, d.$$

Let  $\varphi = \theta + \varepsilon\psi$  with  $\psi \in M$  and  $\varepsilon \in \mathbb{R}$ . Then for every  $t \in [0, T]$  and  $j \in \{1, \dots, d\}$ , we have

$$\begin{aligned} E_{\theta+\varepsilon\psi}(\xi_t^j) &= E_{\theta+\varepsilon\psi}(\theta_t^j + \varepsilon\psi_t^j) \\ &= E_{\theta+\varepsilon\psi}(\theta_t^j) + \varepsilon\psi_t^j. \end{aligned}$$

This implies that for every  $j = 1, \dots, d$

$$\begin{aligned}
\psi_t^j &= \frac{d}{d\varepsilon}_{/\varepsilon=0} E_{\theta+\varepsilon\psi}(\xi_t^j - \theta_t^j) \\
&= E \left( \frac{d}{d\varepsilon}_{/\varepsilon=0} \exp \left[ \sum_{i=1}^d \left( \int_0^t K_H^{-1}(\theta^i + \varepsilon\psi^i)(s) dW_s^i \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{1}{2} \int_0^t (K_H^{-1}(\theta^i + \varepsilon\psi^i)(s))^2 ds \right) \right] (\xi_t^j - \theta_t^j) \right) \\
&= E_\theta \left( \sum_{i=1}^d \left[ \int_0^t K_H^{-1}(\psi^i)(s) dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s) K_H^{-1}(\theta^i)(s) ds \right] \right. \\
&\quad \left. \times (\xi_t^j - \theta_t^j) \right) \\
&= E_\theta \left( \sum_{i=1}^d \left[ \int_0^t K_H^{-1}(\psi^i)(s) d\widetilde{W}_s^i \right] (\xi_t^j - \theta_t^j) \right) \\
&= E_\theta \left( \left[ \int_0^t K_H^{-1}(\psi^j)(s) d\widetilde{W}_s^j \right] (\xi_t^j - \theta_t^j) \right).
\end{aligned}$$

Applying Cauchy-Schwarz inequality in  $L^2(\Omega, dP_\theta)$ , we obtain that for every  $t \in [0, T]$

$$\begin{aligned}
\|\psi_t\|^2 &= \sum_{j=1}^d (\psi_t^j)^2 \leq \sum_{j=1}^d E_\theta \left( (\xi_t^j - \theta_t^j)^2 \right) E_\theta \left( \left[ \int_0^t K_H^{-1}(\psi^j)(s) d\widetilde{W}_s^j \right]^2 \right) \\
&= \sum_{j=1}^d E_\theta \left[ ((\xi_t^j - \theta_t^j)^2) \int_0^t (K_H^{-1}(\psi^j)(s))^2 ds \right].
\end{aligned}$$

We take for each  $j = 1, \dots, d$ ,  $\psi_t^j = t^{2H}$ . Since  $t \longrightarrow t^{2H}$  is absolutely continuous function, then by (2.6), a simple calculation shows that

$$\begin{aligned}
K_H^{-1}(t^{2H}) &= 2Ht^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} t^{H-\frac{1}{2}} \\
&= \frac{2H\beta(\frac{1}{2}-H, H+\frac{1}{2})}{\Gamma(\frac{1}{2}-H)} t^{H-1/2} \\
&= 2H(\Gamma(\frac{1}{2}+H))t^{H-1/2}.
\end{aligned}$$

It is known that

$$0 \leq \Gamma(z) \leq 1 \quad \text{for every } z \in [1, 2]. \quad (3.9)$$



Combining the facts that  $z\Gamma(z) = \Gamma(z+1)$ ,  $z > 0$ ,  $2H \leq (H + \frac{1}{2})^2$  and (3.9), we obtain

$$dt^{2H} = \|\psi_t\|^2 \leq (\Gamma(\frac{3}{2} + H))^2 E_\theta (\|\xi_t - \theta_t\|^2) \leq E_\theta (\|\xi_t - \theta_t\|^2).$$

Hence, by an integration with respect to  $dt$ , we get

$$R(H, \hat{\theta}) = \frac{T^{2H+1}}{2H+1} \leq E_\theta \int_0^T \|\xi_t - \theta_t\|^2 dt.$$

Therefore (3.8) is satisfied.

**Corollary 1** *The process  $\hat{\theta} = B^H$  is a maximum likelihood estimator of  $\theta$ .*

*Proof:* We have for every  $\psi \in M$

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \exp \left[ \sum_{i=1}^d \int_0^t K_H^{-1}(\hat{\theta}^i + \varepsilon\psi^i)(s) dW_s^i - \frac{1}{2} \int_0^t (K_H^{-1}(\hat{\theta}^i + \varepsilon\psi^i)(s))^2 ds \right] = 0.$$

Hence

$$\sum_{i=1}^d \left( \int_0^t K_H^{-1}(\psi^i)(s) dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s) K_H^{-1}(\hat{\theta}^i)(s) ds \right) = 0.$$

Which implies that for every  $i = 1, \dots, d$

$$E \left( \int_0^t K_H^{-1}(\psi^i)(s) dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s) K_H^{-1}(\hat{\theta}^i)(s) ds \right)^2 = 0.$$

Then, for each  $i = 1, \dots, d$

$$W_t^i = \int_0^t K_H^{-1}(\hat{\theta}^i)(s) ds, \quad t \in [0, T].$$

Therefore by (2.3), we obtain that  $B^H = \hat{\theta}$ .

## 4 Superefficient James-Stein type estimators

The aim of this section is to construct superefficient estimators of  $\theta$  which dominate, under the usual quadratic risk, the natural MLE estimator  $B^H$ . The class of estimators considered here are of the form

$$\delta(B^H)_t = B_t^H + g(B_t^H, t), \quad t \in [0, T] \quad (4.10)$$

where  $g : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$  is a function. The problem turns to find sufficient conditions on  $g$  such that  $\mathcal{R}(\theta, \delta(B^H)) < \infty$  and the risk difference is negative, i.e.

$$\Delta\mathcal{R}(\theta) = \mathcal{R}(\theta, \delta(B^H)) - \mathcal{R}(\theta, B^H) < 0.$$

In the sequel we assume that the function  $g$  satisfies the following assumption:

$$(A) \begin{cases} E_\theta \left[ \int_0^T \|g(B_t^H, t)\|_d^2 dt \right] < \infty, \\ \text{the partial derivatives } \partial_i g^i := \frac{\partial g^i}{\partial x^i}, \quad i = 1, \dots, n \text{ of } g \text{ exist.} \end{cases}$$

Then  $\mathcal{R}(\theta, \delta(B^H)) < \infty$ . Moreover

$$\begin{aligned} \Delta\mathcal{R}(\theta) &= E_\theta \left[ \int_0^T \|B_t^H + g(B_t^H, t) - \theta_t\|_d^2 - \|B_t^H - \theta_t\|_d^2 dt \right] \\ &= E_\theta \left[ \int_0^T \|g(B_t^H, t)\|_d^2 + 2 \sum_{i=1}^d \left( g^i(B_t^H, t)(B_t^{H,i} - \theta_t^i) \right) dt \right]. \end{aligned}$$

In addition,

$$\begin{aligned} &E_\theta \int_0^T \sum_{i=1}^d \left( g^i(B_t^H, t)(B_t^{H,i} - \theta_t^i) \right) dt \\ &= \sum_{i=1}^d \int_0^T (2\pi t^{2H})^{-\frac{d}{2}} \left( \int_{\mathbb{R}^d} g^i(x^1, \dots, x^d, t)(x^i - \theta_t^i) \right. \\ &\quad \left. \times e^{-\frac{\sum_{j=1}^d (x^j - \theta_t^j)^2}{2t^{2H}}} dx^1 \dots dx^d \right) dt \\ &= \sum_{i=1}^d \int_0^T (2\pi t^{2H})^{-\frac{d}{2}} \left( \int_{\mathbb{R}^d} t^{2H} \partial_i g^i(x^1, \dots, x^d, t) \right. \\ &\quad \left. \times e^{-\frac{\sum_{j=1}^d (x^j - \theta_t^j)^2}{2t^{2H}}} dx^1 \dots dx^d \right) dt \\ &= \sum_{i=1}^d \int_0^T (t^{2H} E_\theta \partial_i g^i(B_t^H, t)) dt = E_\theta \left[ \sum_{i=1}^d \int_0^T t^{2H} \partial_i g^i(B_t^H, t) dt \right]. \end{aligned}$$

Consequently, the risk difference equals

$$\Delta\mathcal{R}(\theta) = E_\theta \left[ \int_0^T \left( \|g(B_t^H, t)\|_d^2 + 2t^{2H} \sum_{i=1}^d \partial_i g^i(B_t^H, t) \right) dt \right]. \quad (4.11)$$

We can now state the following theorem.

**Theorem 2** *Let  $g : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$  be a function satisfying (A). A sufficient conditions for the estimator  $(B_t^H + g(B_t^H, t))_{t \in [0, T]}$  to dominate  $B^H$  under the usual quadratic risk is*

$$E_\theta \left[ \int_0^T \left( \|g(B_t^H, t)\|^2 + 2t^{2H} \sum_{i=1}^d \partial_i g^i(B_t^H, t) \right) dt \right] < 0.$$

As an application, take  $g$  of the form

$$g(x, t) = at^{2H} \frac{r(\|x\|^2)}{\|x\|^2} x, \quad (4.12)$$

where  $a$  is a constant strictly positive and  $r : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is bounded derivable function. The next lemma give a sufficient condition for  $g$  in (4.12) to satisfies the assumption (A).

**Lemma 1** *If  $d \geq 3$  and  $H < \frac{1}{2}$  then*

$$E \left[ \int_0^T \frac{1}{\|B_t^H\|^2} dt \right] < \infty. \quad (4.13)$$

*Proof:* Firstly the integral given by (4.13) is well defined, because

$$(dt \times P) ((t, w); B_t^H(w) = 0) = 0$$

where  $(dt \times P)$  is the product measure.

Using the change of variable and  $d \geq 3$  we see that

$$E \int_0^T \frac{1}{\|B_t^H\|^2} dt = \int_0^T \frac{dt}{t^{2H}} \int_{\mathbb{R}^d} \frac{e^{-\frac{\|y\|^2}{2}}}{\sqrt{2\pi}\|y\|^2} dy \leq C \int_0^T \frac{1}{t^{2H}} dt,$$

where  $C$  is a constant depending only on  $d$ . Furthermore, since  $H < \frac{1}{2}$  then (4.13) holds.

**Theorem 3** *Assume that  $d \geq 3$ . If the function  $r$ , the constant  $a$  and the parameter  $H$  satisfy:*

i)  $0 \leq r(\cdot) \leq 1$

ii)  $r(\cdot)$  is differentiable and increasing

iii)  $0 < a \leq 2(d-2)$  and  $H < 1/2$ ,

then the estimator

$$\delta(B^H) = B_t^H - at^{2H} \frac{r(\|B_t^H\|^2)}{\|B_t^H\|^2} B_t^H, \quad t \in [0, T].$$

dominates  $B^H$ .

*Proof:* It suffices to prove that  $\Delta\mathcal{R}(\theta) < 0$ . From (4.11) and the hypothesis i) and ii) we can write

$$\begin{aligned} \Delta\mathcal{R}(\theta) &= aE_\theta \left[ \int_0^T t^{4H} \left( \frac{ar^2(\|B_t^H\|^2)}{\|B_t^H\|^2} - 2(d-2) \frac{r(\|B_t^H\|^2)}{\|B_t^H\|^2} \right. \right. \\ &\quad \left. \left. - 4r'(\|B_t^H\|^2) \right) dt \right] \\ &\leq a[a - 2(d-2)] E_\theta \left[ a \int_0^T t^{4H} \frac{r(\|B_t^H\|^2)}{\|B_t^H\|^2} \right]. \end{aligned}$$

Combining this fact with the assumption iii) yields that the risk difference is negative. Which proves the desired result.

For  $r = 1$ , we obtain a James-Stein type estimator:

**Corollary 2** *Let  $d \geq 3$ ,  $0 < H < \frac{1}{2}$  and  $0 < a \leq 2(d-2)$ . Then the estimator*

$$\left( 1 - \frac{at^{2H}}{\|B_t^H\|^2} \right) B_t^H, \quad t \in [0, T]$$

dominates  $B^H$ .

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